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# THE SUBGROUPS OF ORDER A POWER OF 2 OF THE SIMPLE QUINARY ORTHOGONAL GROUP IN THE GALOIS FIELD OF ORDER $p^n = 8l \pm 3$*

BY

LEONARD EUGENE DICKSON

1. The group of all quinary orthogonal substitutions of determinant unity in the $GF[p^n]$, $p > 2$, has a subgroup $O_\Omega$ of index 2 which is simple. The latter is simply isomorphic with the quotient-group $Q$ of the quaternary abelian group and the group composed of the identity and the substitution which merely changes the sign of each variable. The difficulty in the employment of $Q$ is apparent, while for $O_\Omega$ there is unfortunately no known practical† criterion to distinguish its substitutions from the remaining quinary orthogonal substitutions. While the abelian form seems best adapted to the determination‡ of the subgroups of order a power of $p$, the orthogonal form is found to possess advantages in the study of the subgroups of order a power of 2.

The case $p^n = 8l \pm 3$, namely, that in which 2 is a not-square in the $GF[p^n]$, is here treated on account of its simplicity (compare in particular §§ 2, 4, 5, 22) and in view of the applications to be made in subsequent papers in these Transactions to the determination of all the subgroups when $p^n = 3$ and $p^n = 5$.

There is established the remarkable result that, independent of the values of $p$ and $n$ (such that $p^n$ is of the form $8l \pm 3$), the group $O_\Omega$ contains the same number of distinct sets of conjugate subgroups of order each power of 2, one set of representatives serving for every $O_\Omega$ (compare the diagrammatic summary in § 21, the group notations being given in earlier sections in display formulæ separately numbered). Moreover, except for the subgroups of orders 2, 4, and certain types of order 8, the order of the largest subgroup of $O_\Omega$ in which a group of order a power of 2 is self-conjugate is independent of $p$ and $n$.

* Presented to the Society at the Boston meeting, August 31–September 1, 1903. Received for publication, July 28, 1903.

† In the theory we have recourse to the generators (see § 2). When this becomes impracticable, we resort to the isomorphism with the abelian group by means of the "second-compound" theory (compare §§ 11, 40, 44).

‡ Transactions, vol. 4 (1903), pp. 371–386.

By way of check, it may be stated that the results of §§ 10, 11 and all after § 21 were first established by other methods in the case $p^n = 3$ and in part for $p^n = 5$.

ORIENTATION OF THE CASE $p^n = 8l \pm 3$, §§ 2–5.

2. The simple quinary orthogonal group $O_\Omega$ in the $GF[p^n]$, $p > 2$, has the order

$$\Omega_{n,p} = \tfrac{1}{2} p^{4n}(p^{4n} - 1)(p^{2n} - 1). \tag{1}$$

We observe the following lowest orders:

$$\Omega_{1,3} = 2^6 \cdot 3^4 \cdot 5, \qquad \Omega_{1,5} = 2^6 \cdot 3^2 \cdot 5^4 \cdot 13, \qquad \Omega_{1,7} = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4,$$

$$\Omega_{1,11} = 2^6 \cdot 3^2 \cdot 5^2 \cdot 11^4 \cdot 61, \quad \Omega_{1,13} = 2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17,$$

$$\Omega_{1,17} = 2^{10} \cdot 3^4 \cdot 5 \cdot 17^4 \cdot 29, \quad \Omega_{1,19} = 2^6 \cdot 3^4 \cdot 5^2 \cdot 19^4 \cdot 181, \quad \Omega_{2,3} = 2^8 \cdot 3^8 \cdot 5^2 \cdot 41,$$

$$\Omega_{2,5} = 2^8 \cdot 3^2 \cdot 5^8 \cdot 13 \cdot 313, \quad \Omega_{2,7} = 2^{10} \cdot 3^2 \cdot 5^4 \cdot 7^8 \cdot 1201, \quad \Omega_{3,3} = 2^6 \cdot 3^{12} \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 73.$$

Let $p^n = 2k + 1$. Then $\frac{1}{2}(p^{2n} + 1)$ is odd, while

$$(p^{2n} - 1)^2 = 2^6 [\tfrac{1}{2} k(k+1)]^2.$$

Hence $\Omega_{n,p}$ is always divisible by $2^6$. The condition that $2^6$ shall be the highest power of 2 occurring as a factor is that $\frac{1}{2}k(k+1)$ shall be odd. According as $k = 2t$ or $k = 2t - 1$, we have $k = 4j + 2$ or $k = 4j + 1$, upon replacing the odd number $t$ by $2j + 1$. Hence $p^n = 8j + 5$ or $8j + 3$, respectively.

THEOREM. *The highest power of* 2 *occurring as a factor of* $\Omega_{n,p}$ *is* $2^6$ *if and only if* $p^n = 8l \pm 3$.

3. By *Linear Groups*, §§ 181. 182, 189, $O_\Omega$ is generated by

$$Q_{i,j}^{\alpha,\beta} \equiv (O_{i,j}^{\alpha,\beta})^2, \qquad O_{i,j}^{\rho,\sigma} O_{k,l}^{\rho,\sigma} \qquad (i,j,k,l = 1, \cdots, 5), \tag{2}$$

where $\alpha$ and $\beta$ are arbitrary solutions of $x^2 + y^2 = 1$, $\rho$ and $\sigma$ fixed solutions,

$$O_{i,j}^{\alpha,\beta}: \qquad \xi_i' = \alpha\xi_i + \beta\xi_j, \qquad \xi_j' = -\beta\xi_i + \alpha\xi_j \qquad (\alpha^2 + \beta^2 = 1),$$

the cases $p^n = 3$ and $p^n = 5$ alone being exceptional. Let

$$(\xi_i \xi_j): \qquad \xi_i' = \xi_j, \qquad \xi_j' = \xi_i,$$

noting that these *linear* substitutions do not compound as *literal* substitutions; for example, $(\xi_1\xi_3)(\xi_1\xi_2) = (\xi_1\xi_2\xi_3)$. Let

$$C_i: \qquad \xi_i' = -\xi_i, \qquad \xi_j' = \xi_j \qquad (j = 1, \cdots, 5; j \neq i).$$

Then for $p^n = 3$, the generators are the $C_iC_j$, $(\xi_i\xi_j)(\xi_k\xi_l)$, and

$$W = W^{-2}: \qquad \begin{aligned} \xi_1' &= \xi_1 - \xi_2 - \xi_3 - \xi_4, & \xi_2' &= \xi_1 - \xi_2 + \xi_3 + \xi_4, \\ \xi_3' &= \xi_1 + \xi_2 - \xi_3 + \xi_4, & \xi_4' &= \xi_1 + \xi_2 + \xi_3 - \xi_4. \end{aligned}$$

For $p^n = 5$, the generators are the $C_iC_j, (\xi_i\xi_j)(\xi_k\xi_l)$, and

$$R = R^{-1}: \qquad \xi_1' = \xi_1 + \xi_2 + 2\xi_3, \quad \xi_2' = \xi_1 + 2\xi_2 + \xi_3, \quad \xi_3' = 2\xi_1 + \xi_2 + \xi_3.$$

4. The conditions that $Q_{i,j}^{\alpha,\beta}$ shall reduce to $(\xi_i\xi_j)C_i$ are

$$2\alpha^2 = 1, \qquad 2\alpha\beta = -1,$$

solutions of which exist in the $GF[p^n], p > 2$, if and only if 2 is a square. Now 2 is a quadratic residue of all primes of the form $8k \pm 1$ and a quadratic non-residue of all primes $8k \pm 3$. Hence (*Linear Groups*, § 62), 2 is a not square in the $GF[p^n], p > 2$, if and only if $p^n$ is of the form $8l \pm 3$.

THEOREM. *The second type of generators* (2) *may be replaced by* $(\xi_i\xi_j)(\xi_k\xi_l)$ *if and only if* $p^n = 8l \pm 3$.

5. We are therefore led to the group * merely permuting $\xi_1^2, \cdots, \xi_5^2$; viz.,

$$G_{960} = \{\text{group generated by all the } C_iC_j \text{ and } (\xi_i\xi_j)(\xi_k\xi_l)\}. \tag{3}$$

For brevity set $C_0 = C_1C_2C_3C_4C_5$. Then $G_{960}$ has the commutative subgroup

$$G_{16} = \{I,\ C_iC_j\ (i, j = 0, 1, 2, 3, 4, 5;\ j > i)\}. \tag{4}$$

The alternating group on 5 letters is simply isomorphic with the subgroup

$$G_{60} = \{\text{group generated by all the } (\xi_i\xi_j)(\xi_k\xi_l)\}. \tag{5}$$

Extending the group $G_{16}$ by the substitutions

$$B_1 = \text{identity}, \quad B_2 = (\xi_1\xi_2)(\xi_3\xi_4), \quad B_3 = (\xi_1\xi_3)(\xi_2\xi_4), \quad B_4 = (\xi_1\xi_4)(\xi_2\xi_3),$$

we obtain a subgroup of $G_{960}$ whose substitutions are given uniquely thus:

$$G_{64} = \{B_k,\ B_kC_iC_j\ (k = 1, 2, 3, 4;\ i, j = 0, 1, \cdots, 5;\ j > i)\}. \tag{6}$$

THEOREM. *The subgroups of* $O_\Omega$ *of order the highest power of* 2 *contained in* $\Omega$ *are of order* $2^6$ *and conjugate with* $G_{64}$ *if and only if* $p^n$ *is of the form* $8l \pm 3$; *namely, if* 2 *is a not-square in the* $GF[p^n], p > 2$.

## REPRESENTATIVES OF THE SETS OF CONJUGATE SUBGROUPS OF ORDER A POWER OF 2 WITHIN $O_\Omega$, §§ 6–21.

*Distribution of the substitutions of* $G_{64}$ *into sets of conjugates.*

6. The substitutions in the four following sets

$$I,\ C_1C_3,\ C_2C_4,\ C_1C_2C_3C_4; \qquad C_1C_5,\ C_3C_5,\ C_1C_2C_4C_5,\ C_2C_3C_4C_5;$$
$$C_2C_5,\ C_4C_5,\ C_1C_2C_3C_5,\ C_1C_3C_4C_5; \qquad C_1C_2,\ C_1C_4,\ C_2C_3,\ C_3C_4;$$

transform $B_2$ into $B_3, B_3C_1C_3, B_3C_2C_4, B_3C_1C_2C_3C_4$, respectively. Further,

* Two sets of generational relations for $G_{960}$ are given in *Linear Groups*, p. 293.

the $B_i$ are commutative. Hence $B_3$ is conjugate within $G_{64}$ only with $B_3$, $B_3C_1C_3$, $B_3C_2C_4$ and $B_3C_1C_2C_3C_4$. Now $(\xi_3\xi_k)$ transforms $G_{64}$ into itself if $k=2, 3$ or $4$. Hence if $l$ and $m$ denote the two integers left in the set $2, 3, 4$ after the exclusion of $k$, the substitutions

$$B_k,\ B_kC_1C_k,\ B_kC_lC_m,\ B_kC_1C_2C_3C_4$$

form a complete set of conjugates within $G_{64}$. Next $B_i$ transforms $C_1C_5$ into $C_iC_5$, so that the substitutions $C_iC_5 (i=1, 2, 3, 4)$ form a complete set of conjugates. Since $B_2$, $B_3$ and $B_4$ transform $C_1C_2$ into $C_1C_2$, $C_3C_4$ and $C_3C_4$, respectively, $C_1C_2$ is conjugate only with itself and $C_3C_4$. Likewise for $C_1C_3$ and $C_2C_4$, for $C_1C_4$ and $C_2C_3$. Evidently $C_1C_2C_3C_4$ is self-conjugate. Hence $B_2$, $B_3$ and $B_4$ transform $B_kC_1C_2$ into $B_kC_1C_2$, $B_kC_3C_4$ and $B_kC_3C_4$, respectively; while the substitutions of $G_{16}$ transform $B_3C_1C_2$ into $B_3C_1C_2$, $B_3C_2C_3$, $B_3C_1C_4$, $B_3C_3C_4$ and transform $B_3C_3C_4$ into $B_3C_3C_4$, $B_3C_1C_4$, $B_3C_2C_3$, $B_3C_1C_2$. Hence $B_3C_1C_3$ is conjugate only with itself and $B_3C_1C_4$, $B_3C_3C_2$, $B_3C_3C_4$. Applying the above transformation $(\xi_3\xi_k)$, we obtain the conjugates to $B_kC_1C_l$.

Since $B_3$ is one of four conjugates and since $B_i$ transforms $B_3C_1C_5$ into $B_3C_iC_5$, it follows that the substitutions of $G_{64}$ transform $B_3C_1C_5$ only into $B_3C_iC_5$, $B_3C_1C_3C_iC_5$, $B_3C_2C_4C_iC_5$, or $B_3C_1C_2C_3C_4C_iC_5 \equiv B_3C_iC_0$, where $i=1, 2, 3, 4$. Hence $B_3C_iC_5$ is conjugate only with $B_3C_iC_5$ and $B_3C_iC_0 (i=1, 2, 3, 4)$. Applying the transformation $(\xi_3\xi_k)$, we obtain the conjugates to $B_kC_1C_5$.

*The substitutions of $G_{64}$ fall into the following* 16 *distinct sets of conjugates:*

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{C_1C_k,\ C_lC_m\};\ \{C_iC_5 (i=1, 2, 3, 4)\};$$

$$\{B_k,\ B_kC_1C_k,\ B_kC_lC_m,\ B_kC_1C_2C_3C_4\};\ \{C_iC_0 (i=1, 2, 3, 4)\};$$

$$\{B_kC_1C_l,\ B_kC_1C_m,\ B_kC_kC_l,\ B_kC_kC_m\};\ \{B_kC_iC_5,\ B_kC_iC_0 (i=1, 2, 3, 4)\};$$

*where $k=2, 3, 4$, while $l$ and $m$ denote the two integers left in the set* $2, 3, 4$ *after the exclusion of $k$, the order of $l$ and $m$ being immaterial.*

*Determination of all the self-conjugate subgroups of $G_{64}$.*

7. If a self-conjugate subgroup $H$ contains one $C_iC_5$, it contains them all and hence also every $C_iC_j (i, j=1, 2, 3, 4)$, so that $H$ contains $G_{16}$. Similarly, if $H$ contains one $C_iC_0$, it contains $G_{16}$. If $H$ contains $C_1C_k$, or $B_k$, or $B_kC_1C_l$, it contains the respective commutative group

(7) $G_4^k = \{I, C_1C_k, C_lC_m, C_1C_2C_3C_4\}$,

(8) $G_8^k = \{B_i, B_iC_1C_k, B_iC_lC_m, B_iC_1C_2C_3C_4 (i=1, k)\}$,

(9) $H_8^k = \{I, C_1C_k, C_lC_m, C_1C_2C_3C_4, B_kC_1C_l, B_kC_1C_m, B_kC_kC_l, B_kC_kC_m\}$.

If $H$ contains one $B_k C_i C_5$, it contains the group

$$(10)\quad H_{16}^k = \{I,\ C_iC_5,\ C_1C_2C_3C_4,\ B_kC_iC_5,\ B_kC_iC_0 (i,j=1,2,3,4)\}.$$

*Hence the self-conjugate subgroups of $G_{64}$ are given by the series*

$$(11)\quad I,\ G_2 = \{I,\ C_1C_2C_3C_4\},\ G_4^k,\ G_8^k,\ H_8^k,\ G_{16},\ H_{16}^k (k=2,3,4),$$

*together with the groups resulting from the combination of two or more of them.*

Now $G_2$ is a subgroup* of all of order $>2$; while $G_4^k$ is a subgroup of $G_8^k$, $H_8^k$, $G_{16}$, $H_{16}^2$, $H_{16}^3$, $H_{16}^4$. Any two of the groups $G_4^k$ combine into

$$(12)\qquad G_8 = \{I,\ C_iC_j,\ C_1C_2C_3C_4 (i,j=1,2,3,4)\}.$$

Combining $H_8^k$ with either $G_4^l$ or $G_4^m$, we obtain the group

$$(13)\ J_{16}^k = \{I,\ C_iC_j,\ C_1C_2C_3C_4,\ B_k,\ B_kC_iC_j,\ B_kC_1C_2C_3C_4 (i,j=1,2,3,4)\}.$$

The same group results from the combination of $G_8^k$ with either $G_4^l$ or $G_4^m$; also from the combination of $G_8^k$ with $H_8^k$. Combining $H_8^k$ with either $G_8^l$ or $G_8^m$, we get the group of all the substitutions of $G_{64}$ which leave $\xi_5$ fixed:

$$(14)\qquad G_{32} = \{B_t,\ B_tC_iC_j\ B_tC_1C_2C_3C_4 (t,i,j=1,2,3,4)\}.$$

Combining any two of the groups $G_8^2$, $G_8^3$, $G_8^4$, or any two of the groups $H_8^2$, $H_8^3$, $H_8^4$, we obtain $G_{32}$. Combining $G_{16}$ with any one of the groups $G_8^k$, $H_8^k$, $H_{16}^k$, we obtain the group

$$(15)\quad J_{32}^k = \{I,\ C_iC_j,\ B_k,\ B_kC_iC_j (i,j=0,1,2,3,4,5;\ j>i)\}.$$

The same group results from the combination of $H_{16}^k$ with either $G_8^k$ or $H_8^k$. Combining $H_{16}^l$ with either $G_8^k$ or $H_8^k$, we obtain the group

$$(16)\ H_{32}^k = \{I,\ C_iC_j,\ C_1C_2C_3C_4,\ B_k,\ B_kC_iC_j,\ B_kC_1C_2C_3C_4,\ B_tC_iC_5,\ B_tC_iC_0\}$$
$$(i,j=1,2,3,4;\ t=2,3,4;\ t\neq k).$$

We have now combined the groups (11) by pairs in every possible way.

The groups $G_4^2$, $G_4^3$, $G_4^4$, $G_8$ all lie in each of the five new groups (12)–(16), while $G_8$ lies also in $G_{16}$ and $H_{16}^k$. Now $G_8^k$ and $H_8^k$ lie in $J_{16}^k$, $G_{32}$, $J_{32}^k$, $H_{32}^k$, but neither lies in $J_{16}^l$, $J_{32}^l$, $H_{32}^l$. Also $G_{16}$ lies in every $J_{32}^k$, but not in $G_{32}$, nor in any $H_{32}^k$. Finally, $H_{16}^k$ lies in $J_{32}^k$, $H_{32}^l$, $H_{32}^m$, but not in $G_{32}$, $J_{32}^l$, $H_{32}^k$. We have therefore to consider the following compositions:

$$(G_8^k,\ G_8) = (H_8^k,\ G_8) = J_{16}^k, \qquad (G_8^k,\ J_{16}^l) = (H_8^k,\ J_{16}^l) = G_{32},$$
$$(G_8^k,\ J_{32}^l) = (H_8^k,\ J_{32}^l) = G_{64}, \qquad (G_8^k,\ H_{32}^l) = (H_8^k,\ H_{32}^l) = G_{64},$$

* Hence the self-conjugate subgroups may also be determined from a study of the quotient-group $G_{64}/G_2$.

$$(G_{16}, J_{16}^k) = J_{32}^k, \qquad (G_{16}, G_{32}) = (G_{16}, H_{32}^k) = (H_{16}^k, G_{32}) = G_{64},$$

$$(H_{16}^k, J_{16}^k) = J_{32}^k, \quad (H_{16}^k, J_{16}^l) = H_{32}^l, \quad (H_{16}^k, J_{32}^l) = (H_{16}^k, H_{32}^k) = G_{64},$$

noting finally that any two of the groups $G_{32}, J_{32}^k, H_{32}^k, H_{32}^l$ combine into $G_{64}$.

THEOREM. *The group $G_{64}$ contains, in addition to itself, exactly the* 26 *self-conjugate subgroups given by formulæ* (11)–(16).

COROLLARY. *The only subgroups of order* 32 *of $G_{64}$ are*

$$G_{32}, J_{32}^k, H_{32}^k \qquad (k = 2, 3, 4).$$

REMARK. Any three groups marked with the affix $k\,(k = 2, 3, 4)$ are conjugate in $O_\Omega$. No two of the groups $J_{32}^3, H_{32}^3, G_{32}$ are conjugate in $O$ in view of the number of sets of conjugate substitutions in each (§§ 8–10).

*Determination of all the self-conjugate subgroups of $J_{32}^3$.*

8. Proceeding as in § 6, we readily find that the substitutions of $J_{32}^3$ fall into the following 14 distinct sets of conjugates:

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{C_1C_3\};\ \{C_2C_4\};\ \{C_1C_2, C_3C_4\};\ \{C_1C_4, C_2C_3\};$$

$$\{C_1C_5, C_3C_5\};\{C_2C_5, C_4C_5\};\{C_1C_2C_3C_5, C_1C_3C_4C_5\};\{C_1C_2C_4C_5, C_2C_3C_4C_5\};$$

$$\{B_3, B_3C_1C_3, B_3C_2C_4, B_3C_1C_2C_3C_4\};\ \{B_3C_1C_2, B_3C_2C_3, B_3C_1C_4, B_3C_3C_4\};$$

$$\{B_3C_1C_5, B_3C_3C_5, B_3C_1C_0, B_3C_3C_0\};\ \{B_3C_2C_5, B_3C_4C_5, B_3C_2C_0, B_3C_4C_0\}.$$

If a self-conjugate subgroup $H$ contains $C_1C_2$ or $C_1C_4$, it contains the group $G_4^2$ or the group $G_4^4$, respectively. If $H$ contains $C_1C_5$ or $C_2C_5$, it contains one or the other of the commutative groups

$$(17)\quad K_4 = \{I, C_1C_5, C_3C_5, C_1C_3\}, \qquad K_4' = \{I, C_2C_5, C_4C_5, C_2C_4\}.$$

If $H$ contains $C_1C_2C_3C_5$ or $C_1C_2C_4C_5$, it contains one or the other of

$$(18)\quad \begin{aligned} K_4'' &= \{I, C_1C_2C_3C_5, C_1C_3C_4C_5, C_2C_4\}, \\ K_4''' &= \{I, C_1C_2C_4C_5, C_2C_3C_4C_5, C_1C_3\}. \end{aligned}$$

If $H$ contains $B_3$, it contains $G_8^3$. If $H$ contains $B_3C_1C_2$, it contains $H_8^3$. If $H$ contains $B_3C_1C_5$ or $B_3C_2C_5$, it contains the respective commutative group:

$$(19)\quad K_8 = \{I, C_1C_3, C_2C_4, C_1C_2C_3C_4, B_3C_1C_5, B_3C_3C_5, B_3C_1C_0, B_3C_3C_0\},$$

$$(20)\quad K_8' = \{I, C_1C_3, C_2C_4, C_1C_2C_3C_4, B_3C_2C_5, B_3C_4C_5, B_3C_2C_0, B_3C_4C_0\}.$$

Hence the self-conjugate subgroups of $J_{32}^3$ are given by the series

$$(21)\quad \begin{aligned} &I,\ G_2,\ G_2' = \{I, C_1C_3\},\ G_2'' = \{I, C_2C_4\},\ G_4^2,\ G_4^4, \\ &K_4, K_4', K_4'', K_4''', G_8^3, H_8^3, K_8, K_8', \end{aligned}$$

together with the groups resulting from their composition. Now

$$(G_2, G_2') = (G_2, G_2'') = (G_2', G_2'') = G_4^3.$$

Also, $G_2$ lies in every $G_4^k$, $G_8^k$, $H_8^k$, $K_8$, $K_8'$; $(G_2, K_4)$ and $(G_2, K_4')$ give

$$G_8' = \{I, C_1C_3, C_2C_4, C_1C_2C_3C_4, C_1C_5, C_3C_5, C_1C_0, C_3C_0\}, \tag{22}$$

$$G_8'' = \{I, C_1C_3, C_2C_4, C_1C_2C_3C_4, C_2C_5, C_4C_5, C_2C_0, C_4C_0\}, \tag{23}$$

respectively. Also,

$$(G_2, K_4'') = G_8'', (G_2, K_4''') = G_8', (G_2', G_4^2) = (G_2', G_4^4) = G_8,$$

$$(G_2', K_4') = G_8'', (G_2', K_4'') = G_8'',$$

while $G_2'$ lies in $K_4$, $K_4'''$, $G_8^3$, $H_8^3$, $K_8$, $K_8'$, $G_8'$, $G_8''$, $G_8$. Since

$$C_2C_4 = C_1C_3 \cdot C_1C_2C_3C_4,$$

nothing new results from a combination by $G_2''$. By § 9, the groups $G_4^2$, $G_4^3$, $G_4^4$, $G_8^3$, $H_8^3$ and $G_8$ combine to give only the additional group $J_{16}^3$. Now $G_4^2$, $G_4^4$ or $G_8$ combine with any of the groups $K_4$, $K_4'$, $K_4''$, $K_4'''$, $G_8'$, $G_8''$ to give $G_{16}$. Combining $G_4^2$ or $G_4^4$ with either $K_8$ or $K_8'$, we get $H_{16}^3$. Combining $G_4^3$ with either $K_4$ or $K_4'''$, we get $G_8'$; $G_4^3$ with either $K_4'$ or $K_4''$, we get $G_8''$. Now $G_4^3$ is a subgroup of $K_8$, $K_8'$, $G_8$, $G_8'$ and $G_8''$. Next, $K_4$ with $K_4'$ or $K_4''$ gives $G_{16}$, $K_4'$ or $K_4''$ with $K_4'''$ gives $G_{16}$, $K_4$ with $K_4'''$ gives $G_8'$, $K_4'$ with $K_4''$ gives $G_8''$. Next, $(K_4, G_8^3)$ and $(K_4', G_8^3)$ are respectively

$$G_{16}' = \left\{ \begin{matrix} B_i, B_iC_1C_3, B_iC_2C_4, B_iC_1C_2C_3C_4, \\ B_iC_1C_5, B_iC_3C_5, B_iC_1C_0, B_iC_3C_0 (i = 1, 3) \end{matrix} \right\}, \tag{24}$$

$$G_{16}'' = B_2^{-1} G_{16}' B_2. \tag{25}$$

Also, $K_4''$ with $G_8^3$ gives $G_{16}''$, $K_4'''$ with $G_8^3$ gives $G_{16}'$, $K_4$ and $K_4'$ with $H_8^3$ give

$$H_{16}' = \{I, C_1C_3, C_2C_4, C_1C_2C_3C_4, C_1C_5, C_3C_5, C_1C_0, C_3C_0, B_3C_1C_2, \tag{26}$$
$$B_3C_1C_4, B_3C_2C_3, B_3C_3C_4, B_3C_2C_5, B_3C_4C_5, B_3C_2C_0, B_3C_4C_0\},$$

$$H_{16}'' = B_2^{-1} H_{16}' B_2, \tag{27}$$

respectively. Next, $K_4''$ with $H_8^3$ gives $H_{16}''$, $K_4'''$ with $H_8^3$ gives $H_{16}'$,

$$(K_4, K_8) = (K_4''', K_8) = G_{16}', (K_4', K_8) = (K_4'', K_8) = H_{16}''.$$

Interchanging the subscripts 1 with 2 and 3 with 4, we obtain as the compounds of $K_8'$ with $K_4$, $K_4'$, $K_4''$, $K_4'''$, the groups $H_{16}'$ and $H_{16}''$. Next,

$$(G_8^3, K_8) = G_{16}', (G_8^3, K_8') = G_{16}'', (H_8^3, K_8) = H_{16}'', (H_8^3, K_8') = H_{16}',$$

$$(K_8, K_8') = H_{16}^3, (G_8', G_8^3) = G_{16}', (G_8', H_8^3) = H_{16}', (G_8'', G_8^3) = G_{16}'',$$

and $(G_8'', H_8^3) = H_{16}''$. Finally, a combination of a group of order 16 with a group not a subgroup of it evidently gives $J_{32}^3$.

THEOREM. *The group $J_{32}^3$ contains exactly* 26 *self-conjugate subgroups:*

$$I,\ G_2,\ G_2',\ G_2'',\ G_4^2,\ G_4^3,\ G_4^4,\ K_4,\ K_4',\ K_4'',\ K_4''',\ G_8^3,\ H_8^3,$$

$$K_8,\ K_8',\ G_8,\ G_8',\ G_8'',\ G_{16},\ G_{16}',\ G_{16}'',\ H_{16}',\ H_{16}'',\ J_{16}^3,\ H_{16}^3,\ J_{32}^3.$$

COROLLARY. *There are exactly* 7 *subgroups of order* 16 *of $J_{32}^3$.*

*Determination of all the self-conjugate subgroups of $H_{32}^3$.*

9. Its substitutions fall into the following 11 distinct sets of conjugates:

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{C_1C_2,\ C_3C_4\};\ \{C_1C_3,\ C_2C_4\};\ \{C_1C_4,\ C_2C_3\};$$

$$\{B_3, B_3C_1C_3, B_3C_2C_4, B_3C_1C_2C_3C_4\};\ \{B_3C_1C_2, B_3C_1C_4, B_3C_2C_3, B_3C_3C_4\};$$

$$\{B_2C_1C_5, B_2C_3C_5, B_2C_1C_0, B_2C_3C_0\};\ \{B_2C_2C_5, B_2C_4C_5, B_2C_2C_0, B_2C_4C_0\};$$

$$\{B_4C_1C_5, B_4C_3C_5, B_4C_1C_0; B_4C_3C_0\};\ \{B_4C_2C_5, B_4C_4C_5, B_4C_2C_0, B_4C_4C_0\}.$$

Forming the group generated by each substitution and its conjugates, we get

$$I,\ G_2,\ G_4^2,\ G_4^3,\ G_4^4,\ G_8^3,\ H_8^3,\ H_{16}^2,\ H_{16}^2,\ H_{16}^4,\ H_{16}^4,$$

respectively. Combining two or more of them, we obtain the additional groups

$$G_8,\ J_{16}^3,\ H_{32}^3.$$

THEOREM. *The only self-conjugate subgroups of $H_{32}^3$, aside from itself and the identity, are $G_2$, $G_4^2$, $G_4^3$, $G_4^4$, $G_8^3$, $H_8^3$, $G_8$, $H_{16}^2$, $H_{16}^4$, $J_{16}^3$.*

COROLLARY. *There are exactly* 3 *subgroups of order* 16 *in $H_{32}^3$.*

*The self-conjugate subgroups of $G_{32}$.*

10. Its substitutions fall into exactly 17 distinct sets of conjugates. Indeed, aside from the self-conjugate substitutions $I$ and $C_1C_2C_3C_4$, any substitution $S$ is conjugate only with itself and $SC_1C_2C_3C_4$. Now every substitution of $G_{32}$ is of period 2 except identity and the following 12:

$$B_kC_1C_i, \qquad B_kC_kC_i \qquad (k, l = 2, 3, 4;\ k \neq l),$$

the square of any one of which is $C_1C_2C_3C_4$. It follows that, if $S$ ranges over a set of 15 substitutions obtained by taking one and only one of each pair of conjugates within $G_{32}$, the groups

$$(28)\quad I,\ G_2 = \{I,\ C_1C_2C_3C_4\};\ K_4^S = \{I,\ C_1C_2C_3C_4,\ S,\ SC_1C_2C_3C_4\},$$

together with the groups resulting from their composition, give all the self-conjugate subgroups of $G_{32}$.

It is more convenient to proceed by a different method. From what precedes, the quotient-group $Q_{16} = G_{32}/G_2$ is a commutative group all of whose operators, aside from the identity, are of period 2. The quotient of

$$(16-1)(16-2)(16-4) \qquad \text{by} \qquad (8-1)(8-2)(8-4)$$

gives 15 as the number of subgroups of order 8 of $Q_{16}$. Likewise, it contains 35 subgroups of order 4 and 15 of order 2. To every self-conjugate subgroup of $G_{32}$, necessarily containing $C_1C_2C_3C_4$ (as shown above), there corresponds an unique subgroup of $Q_{16}$, and inversely. We may thus readily obtain all the self-conjugate subgroups of $G_{32}$. Those of orders 1, 2, 4 are given by (28). We desire in particular those of order 16.

Denote by $a, b, c, d$ a set of generators of $Q_{16}$. As generators of its 15 subgroups of order 8, we may take

$$(a, b, c);\ (a, b, d);\ (a, c, d);\ (b, c, d);\ (a, b, cd);$$

$$(a, c, bd);\ (a, d, bc);\ (b, c, ad);\ (b, d, ac);\ (c, d, ab);$$

$$(a, bd, cd);\ (b, ad, cd);\ (c, ad, bd);\ (d, ac, bc);\ (ad, bd, cd).$$

For the generators of $Q_{16}$ we may take

$$a = C_1C_2, \qquad b = C_1C_3, \qquad c = B_3, \qquad d = B_2,$$

understanding in this section that $S$ and $SC_1C_2C_3C_4$ are identical operators.

The *analytic* substitution $(\xi_1\xi_3\xi_2)$ transforms the group $(a, b, c)$ into

$$(C_2C_3,\ C_1C_2,\ B_2) = (ab,\ a,\ d) = (a,\ b,\ d).$$

Likewise, $(\xi_1\xi_3\xi_4)$ transforms $(a, b, c)$ into

$$(C_4C_2,\ C_4C_1,\ B_4) = (b,\ ab,\ cd) = a,\ b,\ cd).$$

As shown in § 11, $G_\Omega$ contains a substitution $\Sigma$ which transforms

$$C_1C_2,\ C_1C_4,\ B_2,\ B_3C_1C_4 \qquad \text{into} \qquad B_4C_2C_3,\ B_3C_2C_4,\ C_2C_3,\ B_3C_1C_4,$$

respectively. Hence $\Sigma$ transforms $a$ into $abcd$, $b$ into $ad$, $c$ into $a$, $d$ into $ab$.

It follows that $\Sigma$ transforms $(a, b, d)$ into $(abcd, ad, ab)$, identical with $(ad, bd, cd)$, and transforms the latter into $(cd, bd, b) = (b, c, d)$. Again, $\Sigma$ transforms $(a, b, c)$ into $(a, d, bc)$, and the latter into $(c, d, ab)$. Also, $\Sigma$ transforms $(a, b, cd)$ into $(b, c, ad)$, and the latter into $(a, c, d)$.

Hence the following 9 groups are conjugate within $G_\Omega$:

$$(a, b, c),\ (a, b, d),\ (a, b, cd),\ (ad, bd, cd),\ (b, c, d),$$

$$(a, d, bc),\ (c, d, ab),\ (b, c, ad),\ (a, c, d).$$

It is next shown that the remaining 6 subgroups are conjugate. Now $C_1C_5$, which transforms $B_i$ into $B_iC_1C_i$, transforms

$$(a, c, bd) \qquad \text{into} \qquad (a, cb, bda) = (a, bd, cd).$$

But $\Sigma$ transforms $(a, bd, cd)$ into $(b, d, ac)$, and the latter into $(c, ad, bd)$. Again, $C_1C_5$ transforms $(c, ad, bd)$ and $(b, d, ac)$ into respectively

$$(B_3C_1C_3,\ B_2,\ B_2C_2C_3) = (d, ac, bc),$$

$$(C_1C_3,\ B_2C_1C_3,\ B_3C_1C_4) = (b, ad, cd).$$

To the representatives $(a, b, c)$ and $(a, c, bd)$ of the two sets of conjugate subgroups of $G_{16}$, we adjoin $C_1C_2C_3C_4$ and obtain respectively

$$(C_1C_2,\ C_1C_3,\ B_3,\ C_1C_2C_3C_4)$$
$$= \{B_t,\ B_tC_iC_j,\ B_tC_1C_2C_3C_4\ (i, j = 1, 2, 3, 4;\ t = 1, 3)\},$$
$$(C_1C_2,\ B_3,\ C_1C_3B_2,\ C_1C_2C_3C_4) = F_{16},$$

the former being $J_{16}^3$ and the latter defined as follows:

$$(29) \qquad F_{16} = \left\{ \begin{matrix} B_t,\ B_tC_1C_2,\ B_tC_3C_4,\ B_tC_1C_2C_3C_4, \\ B_iC_1C_3,\ B_iC_2C_3,\ B_iC_1C_4,\ B_iC_2C_4\ (t = 1, 3;\ i = 2, 4) \end{matrix} \right\}.$$

THEOREM. *Within $O_\Omega$ the* 15 *subgroups of order* 16 *of $G_{32}$ are conjugate with the groups $J_{16}^3$ and $F_{16}$, the latter being not conjugate* (§ 13).

11. THEOREM. *The group $O_\Omega$ contains one and but one substitution of period* 3 *which transforms $B_3C_1C_4$ into itself and transforms $C_1C_4$, $C_1C_2$, $B_2$ into $B_3C_2C_4$, $B_4C_2C_3$, $C_2C_3$, respectively.*

If $S$ is commutative with $(B_3C_1C_4)^2 = C_1C_2C_3C_4$, it replaces $\xi_5$ by $\pm\xi_5$ (§ 25). Denoting the matrix of $S$ by $(\alpha_{ij})$, we find that $B_3C_1C_4S = SB_3C_1C_4$ leads to the conditions:

$$\alpha_{31} = -\alpha_{13},\ \alpha_{32} = \alpha_{14},\ \alpha_{33} = \alpha_{11},\ \alpha_{34} = -\alpha_{12},\ \alpha_{41} = \alpha_{23},\ \alpha_{42} = -\alpha_{24},$$
$$\alpha_{43} = -\alpha_{21},\ \alpha_{44} = \alpha_{22}.$$

Hence $S$ is commutative with $B_3C_1C_4$ if and only if it has the form

$$S' = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & 0 \\ -\alpha_{13} & \alpha_{14} & \alpha_{11} & -\alpha_{12} & 0 \\ \alpha_{23} & -\alpha_{24} & -\alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{pmatrix}.$$

The conditions for $C_1C_4S' = S'B_3C_2C_4$ are

$$\alpha_{13} = \alpha_{11}, \qquad \alpha_{14} = \alpha_{12}, \qquad \alpha_{23} = \alpha_{21}, \qquad \alpha_{24} = \alpha_{22}.$$

The conditions for $C_1C_2S' = S'B_4C_2C_3$ and $B_2S' = S'C_2C_3$ then reduce to

$$\alpha_{12} = \alpha_{11}, \qquad \alpha_{21} = -\alpha_{11}, \qquad \alpha_{22} = \alpha_{11}.$$

The resulting substitution is orthogonal if and only if $4\alpha_{11}^2 = 1$. Its determinant is $\pm 16\alpha_{11}^4$. Hence must $\pm 1$ equal $+1$. With these conditions satisfied, $S' = S'^{-2}$ if and only if $\alpha_{11} = -\frac{1}{2}$. Then $S'$ becomes

$$\Sigma = \begin{Vmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

It has been shown that $\Sigma$ belongs to the group of all orthogonal substitutions of determinant unity. It remains to show that $\Sigma$ belongs to $O_\Omega$. For $p^n = 3$, $\Sigma = W^2(\xi_2\xi_3\xi_4)$ and hence is in $O_\Omega$. For $p^n = 5$,

$$\Sigma = C_3C_4(\xi_2\xi_4\xi_3)R_{234}C_3C_5R_{124}R_{312}C_2C_5(\xi_1\xi_4\xi_2),$$

and hence belongs to $O_\Omega$. For $p^n = 11$, we find that

$$\Sigma = O_{1,3}^{5,-3}O_{1,2}^{5,-3}(\xi_1\xi_3\xi_4)O_{1,4}^{5,-3}O_{1,3}^{5,-3}(\xi_1\xi_3\xi_4)C_3C_4(O_{2,3}^{5,3}O_{2,4}^{5,3})^2(\xi_2\xi_4\xi_3)C_2C_4,$$

and hence belongs to $G_\Omega$.

We next treat the general case in which $-1$ is the square of a mark $i$ of the $GF[p^n]$, proceeding as in *Linear Groups*, pp. 179–180. Making the transformation of variables there defined, we find that $\Sigma$ becomes

| | $Y_{12}$ | $Y_{13}$ | $Y_{14}$ | $Y_{23}$ | $Y_{24}$ | $Y_{34}$ |
|---|---|---|---|---|---|---|
| $Y'_{12} =$ | $1/4$ | $(1+i)/4$ | $-i/4$ | $-i/4$ | $(1-i)/4$ | $3/4$ |
| $Y'_{13} =$ | $(-1+i)/4$ | $(-1-i)/2$ | $(1-i)/4$ | $(1-i)/4$ | $0$ | $(1-i)/4$ |
| $Y'_{14} =$ | $-i/4$ | $(-1-i)/4$ | $1/4$ | $-3/4$ | $(1-i)/4$ | $i/4$ |
| $Y'_{23} =$ | $-i/4$ | $(-1-i)/4$ | $-3/4$ | $1/4$ | $(1-i)/4$ | $i/4$ |
| $Y'_{24} =$ | $(-1-i)/4$ | $0$ | $(-1-i)/4$ | $(-1-i)/4$ | $(-1+i)/2$ | $(1+i)/4$ |
| $Y'_{34} =$ | $3/4$ | $(-1-i)/4$ | $i/4$ | $i/4$ | $(-1+i)/4$ | $1/4$ |

This substitution is found to be the second compound of

$$\begin{bmatrix} (1-i)/4 & (1-i)/4 & (3+i)/4 & (-1+i)/4 \\ (-1-i)/4 & (1+i)/4 & (1+i)/4 & (3-i)/4 \\ (3+i)/4 & (-1+i)/4 & (1-i)/4 & (1-i)/4 \\ (1+i)/4 & (3-i)/4 & (-1-i)/4 & (1+i)/4 \end{bmatrix},$$

which is a special abelian substitution. Hence $\Sigma$ belongs to $G_\Omega$.

*Determination of all the self-conjugate subgroups of* $J_{16}^3$.

12. Its substitutions fall into the following 10 distinct sets of conjugates:

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{C_1C_3\};\ \{C_2C_4\};\ \{C_1C_2,\ C_3C_4\};\ \{C_1C_4,\ C_2C_3\};$$

$$\{B_3, B_3C_1C_2C_3C_4\};\{B_3C_1C_2, B_3C_3C_4\};\{B_3C_1C_3, B_3C_2C_4\};\{B_3C_1C_4, B_3C_2C_3\}.$$

The only substitutions of period 4 are $B_3C_1C_2$, $B_3C_3C_4$, $B_3C_1C_4$, $B_3C_2C_3$.

The self-conjugate subgroups of $J_{16}^3$ are

$$I,\ G_2,\ G_2',\ G_2'',\ G_4^1,\ G_4^S \qquad (S = B_3,\ B_3C_1C_2,\ B_3C_1C_3,\ B_3C_1C_4) \tag{30}$$

together with all their combinations. Now $G_2$ lies in all these groups of order $>2$. As shown in §§ 7–8, the groups $G_2$, $G_2'$, $G_2''$, $G_4^2$, $G_4^1$ combine to give only the additional groups $G_4^3$ and $G_8$. Either $G_2'$ or $G_2''$ combines with $K_4^S$ for $S = B_3$ or $B_3C_1C_3$ to give $G_8^3$. Either $G_2'$ or $G_2''$ combines with $K_4^S$ for $S = B_3C_1C_2$ or $B_3C_1C_4$ to give $H_8^3$. Combining $K_4^S$ and $K_4^{S'}$ for the following pairs

$$(S, S') = (B_3, B_3C_1C_3), (B_3C_1C_2, B_3C_1C_4), (B_3, B_3C_1C_2),$$
$$(B_3, B_3C_1C_4), (B_3C_1C_2, B_3C_1C_3), (B_3C_1C_3, B_3C_1C_4),$$

we get the respective groups $G_8^3$, $H_8^3$, $J_8$, $J_8'$, $J_8''$, $J_8'''$, where

$$J_8 = \{I,\ C_1C_2,\ C_3C_4,\ C_1C_2C_3C_4,\ B_3,\ B_3C_1C_2,\ B_3C_3C_4,\ B_3C_1C_2C_3C_4\}, \tag{31}$$

$$J_8' = \{I,\ C_1C_4,\ C_2C_3,\ C_1C_2C_3C_4,\ B_3,\ B_3C_1C_4,\ B_3C_2C_3,\ B_3C_1C_2C_3C_4\}, \tag{32}$$

$$J_8'' = \{I,\ C_1C_4,\ C_2C_3,\ C_1C_2C_3C_4,\ B_3C_1C_2,\ B_3C_1C_3,\ B_3C_2C_4,\ B_3C_3C_4\}, \tag{33}$$

$$J_8''' = \{I,\ C_1C_2,\ C_3C_4,\ C_1C_2C_3C_4,\ B_3C_1C_3,\ B_3C_1C_4,\ B_3C_2C_3,\ B_3C_2C_4\}, \tag{34}$$

each of the groups $J$ being non-commutative. Finally $G_4^2$ combines with the four $K_4^S$, in order, to give $J_8$, $J_8'$, $J_8'''$, $J_8''$; while $G_4^1$ combines with them to give $J_8'$, $J_8''$, $J_8'''$, $J_8$.

THEOREM. *The self-conjugate subgroups of* $J_{16}^3$ *are the groups* (30)–(34), *together with* $G_4^3$, $G_8$, $G_8^3$, $H_8^3$, $J_{16}^3$.

COROLLARY. *The only subgroups of order* 8 *of* $J_{16}^3$ *are* $G_8$, $G_8^3$, $H_8^3$, $J_8$, $J_8'$, $J_8''$, $J_8'''$, *of which the first three only are commutative groups.*

*Determination of all the self-conjugate subgroups of* $F_{16}$.

13. Its substitutions fall into the 10 distinct sets of conjugates

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{B_2C_1C_3\};\ \{B_2C_2C_4\};\ \{C_1C_2,\ C_3C_4\};\ \{B_2C_2C_3,\ B_2C_1C_4\};$$

$$\{B_3,\ B_3C_1C_2C_3C_4\};\ \{B_3C_1C_2,\ B_3C_3C_4\};\ \{B_4C_1C_3,\ B_4C_2C_4\};\ \{B_4C_1C_4,\ B_4C_2C_3\}.$$

Since $B_2C_1C_3$ is of period 4, it follows that $F_{16}$ and $J_{16}^3$ are not isomorphic.

The self-conjugate subgroups of $F_{16}$ are the groups

$$(35)\qquad I,\ G_2,\ C_4=(B_2C_1C_3),\ G_4^2,\ K_4^S\quad (S=B_2C_2C_3,\ B_3,\ B_3C_1C_2,\ B_4C_1C_2,\ B_4C_1C_4),$$

together with all their combinations. Now $G_2$ lies in all those of order 4. Combining $C_4$ with the last six groups (35) in turn, we get the commutative groups $H_8^2$, $H_8^2$, $F_8$, $F_8'$, $F_8$, $F_8'$, where

$$(36)\ F_8=\{I,\ C_1C_2C_3C_4,\ B_3,\ B_3C_1C_2C_3C_4,\ B_iC_1C_3,\ B_iC_2C_4(i=2,\ 4)\},$$

$$(37)\ F_8'=\{I,\ C_1C_2C_3C_4,\ B_2C_1C_3,\ B_2C_2C_4,\ B_3C_1C_2,\ B_3C_3C_4,\ B_4C_1C_4,\ B_4C_2C_3\}.$$

Combining every pair of the $K_4^S$, we get $F_8$, $F_8'$, $J_8$ and $F_8^*$ each one, and $F_8''$ and $F_8'''$ each three times, where

$$(38)\ F_8''=\{I,\ C_1C_2C_3C_4,\ B_3,\ B_3C_1C_2C_3C_4,\ B_iC_1C_4,\ B_iC_2C_3(i=2,\ 4)\},$$

$$(39)\ F_8'''=\{I,\ C_1C_2C_3C_4,\ B_2C_1C_4,\ B_2C_2C_3,\ B_3C_1C_2,\ B_3C_3C_4,\ B_4C_1C_3,\ B_4C_2C_4\},$$

$$(40)\ F_8^*=\{I,\ C_1C_2,\ C_3C_4,\ C_1C_2C_3C_4,\ B_4C_1C_3,\ B_4C_1C_4,\ B_4C_2C_3,\ B_4C_2C_4\}.$$

Finally, $G_4^2$ combines with the $K_4^S$, in order, to give $H_8^2$, $J_8$, $J_8$, $F_8^*$, $F_8^*$.

THEOREM.* *The self-conjugate subgroups of* $F_{16}$ *are the groups* (35)–(40), $H_8^2$, $J_8$, $F_8^*$ *and* $F_{16}$.

COROLLARY. *The group* $F_{16}$ *has exactly* 7 *subgroups of order* 8. *Of them* $H_8^2$, $F_8$ *and* $F_8'$ *are all commutative groups, while* $J_8$, $F_8^*$, $F_8''$ *and* $F_8'''$ *are not.*

*Determination of all the self-conjugate subgroups of* $H_{16}^3$.

14. Its substitutions fall into the 10 distinct sets of conjugates:

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{C_1C_3\};\ \{C_2C_4\};\ \{C_1C_2,\ C_3C_4\};\ \{C_1C_4,\ C_2C_3\};$$

$$\{B_3C_iC_5,\ B_3C_iC_0\}\qquad (i=1,\ 2,\ 3,\ 4).$$

It contains exactly 8 substitutions of period 4:

* Another proof may be based on the quotient-group, $F_{16}/G_2$, which is a commutative group all of whose operators aside from identity are of period 2.

and
$$B_3C_1C_5,\ B_4C_1C_0,\ B_3C_3C_5,\ B_3C_3C_0 \quad \text{(whose squares are } C_1C_3\text{)},$$
$$B_3C_2C_5,\ B_3C_2C_0,\ B_3C_4C_5,\ B_3C_4C_0 \quad \text{(whose squares are } C_2C_4\text{)}.$$

Hence $H_{16}^3$ is not isomorphic with $J_{16}^3$. Having its self-conjugate substitutions all of period 1 or 2, it is not isomorphic with $F_{16}$.

The groups $I,\ G_2,\ G_2',\ G_2'',\ G_4^2,\ G_4^4,\ K_8,\ K_8'$, together with their combinations, give all the self-conjugate subgroups of $H_{16}^3$. Proceeding as in § 8, we find that the only additional groups are $G_4^3,\ G_8,\ H_{16}^3$.

THEOREM. *The only self-conjugate subgroups of $H_{16}^3$, aside from itself and identity, are $G_2,\ G_2',\ G_2'',\ G_4^2,\ G_4^3,\ G_4^4,\ K_8,\ K_8',\ G_8$.*

COROLLARY. *The only subgroups of order 8 of $H_{16}^3$ are $K_8,\ K_8'$ and $G_8$.*

*The fifteen subgroups of order 8 of $G_{16}$.*

15. Since all the substitutions, except identity, of the commutative group $G_{16}$ are of period 2, it contains exactly 15 subgroups of order 8 (see § 10). Since there are but 5 products each of 4 of the $C_i$, any subgroup of order 8 contains at least two $C_iC_j$. Transforming by a suitable even substitution on $\xi_1, \cdots, \xi_5$, we may take $C_1C_3$ as the first generator. Suppose first that there is present at least one further $C_1C_i$ or one $C_3C_j$. Transforming $C_1C_i$ by a suitable power of $(\xi_2\xi_4\xi_5)$, we obtain as first and second generators $C_1C_3$ and $C_1C_2$. The only resulting groups are $G_8$ of § 7 and

$$M_8 = \{I,\ C_1C_3,\ C_1C_2,\ C_2C_3,\ C_1C_5,\ C_3C_5,\ C_2C_5,\ C_1C_2C_3C_5\},$$

$$N_8 = \{I,\ C_1C_3,\ C_1C_2,\ C_2C_3,\ C_4C_5,\ C_1C_3C_4C_5,\ C_1C_2C_4C_5,\ C_2C_3C_4C_5\}.$$

Suppose, however, that there is present no $C_1C_i$ and no $C_3C_j$ other than $C_1C_3$. Then there must occur one of the following three: $C_2C_4,\ C_2C_5,\ C_4C_5$. But $(\xi_2\xi_5\xi_4)$ transforms $C_2C_5$ into $C_4C_2$ while $(\xi_2\xi_4\xi_5)$ transforms $C_4C_5$ into $C_2C_4$. Hence we may take $C_1C_3$ and $C_2C_4$ as the first and second generators. The group does not contain $C_1C_2C_4C_5$ or $C_2C_3C_4C_5$, not having $C_1C_5$ or $C_3C_5$ by assumption. Hence the group can contain only the 8 substitutions forming $G_8''$ of § 8.

Now $(\xi_2\xi_4\xi_5)$ transforms $G_8$ into $M_8$. Also $(\xi_1\xi_2\xi_5\xi_3\xi_4)$ transforms $G_8''$ into $N_8$. Finally, $G_8$, which contains a single product of four $C_i$, is not conjugate under linear transformation with $G_8''$, which contains three products of four $C_i$, since a product of two $C_i$ and a product of four $C_i$ have different characteristic determinants.

THEOREM. *Within $O_\Omega$ every subgroup of order 8 of $G_{16}$ is conjugate with $G_8$ or else with $G_8''$, while the latter are not conjugate.*

*All the self-conjugate subgroups of $G'_{16}$.*

**16.** Its substitutions fall into the 10 distinct sets of conjugates:

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{C_1C_3\};\ \{C_2C_4\};\ \{C_1C_5,\ C_3C_5\};\ \{C_1C_0,\ C_3C_0\};$$

$$\{B_3, B_3C_1C_3\};\ \{B_3C_2C_4, B_3C_1C_2C_3C_4\};\ \{B_3C_1C_5, B_3C_3C_5\};\ \{B_3C_1C_0, B_3C_3C_0\}.$$

The only substitutions of period 4 are the four in the last two sets. Hence $G'_{16}$ is not isomorphic with $H^3_{16}$; also, evidently not with $F_{16}$. Since $B_3C_1C_0$ has the characteristic determinant $(1-\rho)(1+\rho)^2(1+\rho^2)$, while the four substitutions $B_3C_1C_2$, etc., of period 4 in $J^3_{16}$ have the characteristic determinant $(1-\rho)(1+\rho^2)^2$, the groups $G'_{16}$ and $J^3_{16}$ are not conjugate under linear transformation.

The self-conjugate subgroups of $G'_{16}$ are all given by

$$(41)\qquad I,\ G_2,\ G'_2,\ G''_2,\ K_4,\ K'''_4,\ C^5_4 = (B_3C_1C_5),\ C^0_4 = (B_3C_1C_0),$$

$$(42)\qquad \begin{cases} K^*_4 = \{I,\ C_1C_3,\ B_3,\ B_3C_1C_3\}, \\ K^{**}_4 = \{I,\ C_1C_3,\ B_3C_2C_4,\ B_3C_1C_2C_3C_4\}, \end{cases}$$

together with their combinations. Now $G'_2 = \{I, C_1C_3\}$ is a subgroup of all of order 4. By §8, any two of $G_2$, $G'_2$, $G''_2$ generate $G^3_4$, while $G_2$ with either $K_4$ or $K'''_4$ gives $G'_8$. Also $G''_2$ with either $K_4$ or $K'''_4$ gives $G'_8$. Either $G_2$ or $G''_2$ with either $C^5_4$ or $C^0_4$ gives $K_8$. Either $G_2$ or $G''_2$ with either $K^*_4$ or $K^{**}_4$ gives $G^3_8$. Next, $K_4$ with either $C^5_4$ or $K^*_4$ gives

$$(43)\qquad L_8 = \{I,\ C_1C_5,\ C_3C_5,\ C_1C_3,\ B_3,\ B_3C_1C_5,\ B_3C_3C_5,\ B_3C_1C_3\}.$$

Also, $K_4$ with either $C^0_4$ or $K^{**}_4$ gives

$$(44)\qquad L'_8 = \{I,\ C_1C_5,\ C_3C_5,\ C_1C_3,\ B_3C_1C_0,\ B_3C_3C_0,\ B_3C_5C_0,\ B_3C_2C_4\}.$$

Now $K'''_4$ with either $C^0_4$ or $K^*_4$ gives

$$(45)\qquad T_8 = \{I,\ C_1C_3,\ C_1C_0,\ C_3C_0,\ B_3,\ B_3C_1C_3,\ B_3C_1C_0,\ B_3C_3C_0\}.$$

Again, $K'''_4$ with either $C^5_4$ or $K^{**}_4$ gives

$$(46)\qquad T'_8 = \{I,\ C_1C_3,\ C_1C_0,\ C_3C_0,\ B_3C_1C_5,\ B_3C_3C_5,\ B_3C_5C_0,\ B_3C_2C_4\}.$$

Finally, we have the relations

$$(C^5_4,\ C^0_4) = G'_8, \qquad (C^5_4,\ K^*_4) = L_8, \qquad (C^5_4,\ K^{**}_4) = T'_8,$$

$$(C^0_4,\ K^*_4) = T_8, \qquad (C^0_4,\ K^{**}_4) = L'_8, \qquad (K^*_4,\ K^{**}_4) = G^3_8.$$

THEOREM. *The self-conjugate subgroups of $G'_{16}$ are the groups* (41)–(46) *and $G^3_4$, $G'_8$, $K_8$, $G^3_8$, $G'_{16}$.*

COROLLARY. *The subgroups of order* 8 *of $G'_{16}$ are $L_8$, $L'_8$, $T_8$, $T'_8$, $G'_8$, $K_8$, and $G^3_8$.*

*All the self-conjugate subgroups of* $H'_{16}$.

17. Its substitutions fall into the following 10 distinct sets of conjugates:

$$\{I\};\ \{C_1C_2C_3C_4\};\ \{C_1C_3\};\ \{C_2C_4\};\ \{C_1C_5,\ C_3C_5\};\ \{C_1C_0,\ C_3C_0\};$$
$$\{B_3C_1C_2,\ B_3C_2C_3\};\ \{B_3C_1C_4,\ B_3C_3C_4\};\ \{B_3C_2C_5,\ B_3C_4C_0\};$$
$$\{B_3C_4C_5,\ B_3C_2C_0\}.$$

Only the last 8 are of period 4, so that $H'_{16}$ is not isomorphic with $G'_{16}$, $G_{16}$, or $J^3_{16}$. It is not conjugate with $F_{16}$ in view of the periods of their self-conjugate substitutions. Finally, $H'_{16}$ and $H^3_{16}$ are not conjugate * within $O_\Omega$ since they are self-conjugate only under $J^3_{32}$ and $G_{64}$, respectively (§§ 31, 46).

THEOREM. *The only self-conjugate subgroups of* $H'_{16}$ *are* $I$, $G_2$, $G'_2$, $G''_2$, $K_4$, $K'''_4$, $H^3_8$, $K'_8$ *and the groups* $G^3_4$, $G'_4$, $H'_{16}$, *resulting from their combination.*

COROLLARY. *The only subgroups of order* 8 *of* $H'_{16}$ *are* $H^3_8$, $K'_8$, $G'_8$.

*The non-conjugate subgroups of orders* 8, 16, 32 *of* $G_{64}$.

18. There are 3 distinct sets of conjugate subgroups of order 32 in $O_\Omega$, representatives of which are $J^3_{32}$, $H^3_{32}$, $G_{32}$ (end of § 7); 6 distinct sets of order 16, represented by $G_{16}$, $G'_{16}$, $H'_{16}$, $J^3_{16}$, $H^3_{16}$, $F_{16}$ (§§ 8–17). These 6 have only the following subgroups of order 8: $G_8$, $G'_8$, $G''_8$, $G^3_8$, $H^3_8$, $H^2_8$, $J_8$, $J'_8$, $J''_8$, $J'''_8$, $F_8$, $F'_8$, $F''_8$, $F'''_8$, $F^*_8$, $K_8$, $K'_8$, $L_8$, $L'_8$, $T_8$, $T'_8$, together with subgroups of $G_{16}$ conjugate with $G_8$ or $G''_8$ (§§ 12–17).

Now $B_2 \equiv (\xi_1\xi_2)(\xi_3\xi_4)$ transforms $G'_8$ into $G'''_8$, and transforms $K_8$ into $K'_8$; $C_1C_5$ transforms $J_8$ into $J'''_8$, and $J'_8$ into $J''_8$; $(\xi_2\xi_4\xi_3)$ transforms $J''_8$ into $F^*_8$; $\Sigma$ transforms $J_8$ into $F^*_8$, $F_8$ into $H^2_8$, $H^2_8$ into $F'_8$, and $F''_8$ into $J_8$. Finally, $C_2C_5$ transforms $L_8$ into $L'_8$, and $T_8$ into $T'_8$. Hence the above 21 groups are conjugate within $O_\Omega$ with the following:

$$(47) \qquad G_8,\ G''_8,\ G^3_8,\ J_8,\ L_8,\ T_8,\ H^3_8,\ K_8,\ F'''_8.$$

The numbers of substitutions of period 4 in these groups are respectively

$$0,\ 0,\ 0,\ 2,\ 2,\ 2,\ 4,\ 4,\ 6.$$

In the first place, no two of the groups $J_8$, $L_8$, $T_8$, having exactly 2 substitutions of period 4, are conjugate under $O_\Omega$. Indeed, the two $B_3C_1C_2$ and $B_3C_3C_4$ of $J_8$ have the characteristic determinant $(1-\rho)(1+\rho^2)^2$, while the two $B_3C_1C_5$ and $B_3C_3C_5$ of $L_8$ and the two $B_3C_1C_0$ and $B_3C_3C_0$ of $T_8$ all have

* Another proof follows from Lemma I, § 22, taking $t=5$, since $S$ transforms $C_1C_2C_3C_4$ into a substitution of $G_{960}$ only if it replaces some $\xi_r$ by $\pm\xi_5$. Then $r=5$, since $S$ must transform $C_1C_3$ and $C_2C_4$ amongst themselves. Hence $S$ replaces $\xi_5$ by $\pm\xi_5$ and cannot transform $C_1C_2$ or $C_1C_4$ into a substitution involving $\xi_5$.

the characteristic determinant $(1-\rho)(1+\rho)^2(1+\rho^2)$. Moreover, the five of period 2 in $L_8$ all have the characteristic determinant $(1+\rho)^2(1-\rho)^3$, while $C_1C_0 \equiv C_2C_3C_4C_5$ of $T_8$ has $(1+\rho)^4(1-\rho)$.

In the second place, the groups $H_8^3$ and $K_8$ are not conjugate, since the four of period 4 in $H_8^3$ have the characteristic determinant $(1-\rho)(1+\rho^2)^2$, while the four of period 4 in $K_8$ have $(1-\rho)(1+\rho)^2(1+\rho^2)$.

Finally, no two of the groups $G_8$, $G_8''$, $G_8^3$ are conjugate within $O_\Omega$. Indeed all the substitutions except $I$ and $C_1C_2C_3C_4$, of both $G_8$ and $G_8^3$ have the determinant $(1+\rho)^2(1-\rho)^3$, while $C_1C_2C_3C_4$, $C_2C_0$ and $C_4C_0$ of $G_8''$ have the determinant $(1+\rho)^4(1-\rho)$, only four of $G_8''$ having $(1+\rho)^2(1-\rho)^3$. To show that $G_8$ and $G_8^3$ are not conjugate under $O_\Omega$, we note that (§ 34) $G_8$ is self-conjugate only under $G_{192}$ and (§ 32) $G_8^3$ only under $H_{192}$, while $G_{192}$ contains a single subgroup $G_{64}$ of order 64, and $H_{192}$ three subgroups of order 64.

THEOREM. *Within $O_\Omega$ every subgroup of order 8 is conjugate with one and but one of the nine groups* (47).

*The subgroups of order* 4.

19. The commutative group $G_8$ of substitutions of period 2, aside from identity, has exactly 7 subgroups of order 4. Any such subgroup contains at least two $C_iC_j$. Transforming by a suitable even substitution on $\xi_1, \xi_2, \xi_3, \xi_4$, we may take $C_1C_2$ as the first generator. It contains a second $C_iC_j$ of the form $C_2C_3$, $C_2C_4$, or $C_3C_4$, so that the groups are $G_4^2$ or

$$G_4 = \{I,\ C_1C_2,\ C_2C_3,\ C_1C_3\};\quad G_4^* = \{I,\ C_1C_2,\ C_2C_4,\ C_1C_4\}.$$

Now $B_2$ transforms $G_4$ into $G_4^*$, and $(\xi_1\xi_5\xi_4)$ transforms $G_4^*$ into $K_4'$. But $G_4^2$ and $K_4'$ are not conjugate in view of the characteristic determinants of their substitutions.

Each of the 7 subgroups of order 4 of $G_8''$ contains at least one $C_iC_j$. Now $(\xi_2\xi_4\xi_5)$ transforms $C_4C_5$ into $C_2C_4$, while $(\xi_2\xi_5\xi_4)$ transforms $C_2C_5$ into $C_2C_4$, each transforming $G_8''$ into itself. As first generator we may therefore take $C_1C_3$ or $C_2C_4$. The resulting groups are $G_4^3$, $K_4'$, $K_4''$, and

$$G_4' = \{I,\ C_1C_3,\ C_2C_5,\ C_1C_2C_3C_5\},\ G_4'^* = \{I,\ C_1C_3,\ C_4C_5,\ C_1C_3C_4C_5\},$$

the latter being transformed into the former by $B_3$. But $(\xi_1\xi_5\xi_4)$ transforms $G_4'$ into $G_4^2$, while $B_2$ transforms $K_4''$ into $K_4'''$.

The commutative group $G_8^3$ of substitutions of periods 1 and 2 has exactly 7 subgroups of order 4. Now $C_1C_5$, $C_2C_5$, $C_1C_2$, $\Sigma(\xi_2\xi_4\xi_5)$, $\Sigma(\xi_2\xi_4\xi_5)B_2$ transform $G_8^3$ into itself and, in particular, transform $B_3$ into $B_3C_1C_3$, $B_3C_2C_4$, $B_3C_1C_2C_3C_4$, $C_2C_4$, $C_1C_3$, respectively. Hence we may take $C_1C_3$ as the first generator of a subgroup of order 4. The group is therefore $G_4^3$ or else it contains one of the substitutions $B_3$, $B_3C_1C_3$, $B_3C_2C_4$, $B_3C_1C_2C_3C_4$. Now

$I$, $C_1C_5$, $C_2C_5$, $C_1C_2$ transform the preceding four amongst themselves transitively. Hence the resulting groups are conjugate with $K_4^*$ of § 16. Its substitutions, other than identity, have the characteristic determinant $(1-\rho)^3(1+\rho)^2$, so that it is not conjugate with either $G_4^2$ or $K_4'''$. But $K_4^*$ is not conjugate with $K_4'$ by §§ 38, 42.

The group $J_8$ contains a single cyclic group $(B_3C_1C_2)$ of order 4. It remains to determine the groups containing only operators of periods 1 and 2. Since $B_3$ transforms $C_1C_2$ into $C_3C_4$, we may take $C_1C_2$ or $B_3$ as the first generator. The resulting groups are $G_4^2$ and $K_4^{B_3}$ of § 10. The latter is transformed into $G_4^2$ by $\Sigma$.

The group $H_8^3$ contains only two cyclic groups of order 4: $C_4^3=(B_3C_1C_4)$ and $(B_3C_1C_2)$, the latter being transformed into the former by $C_2C_5$. The only further subgroup of order 4 is $G_4^3$.

The group $F_8''''$ has three cyclic groups of order 4: $(B_3C_1C_2)$, $(B_2C_1C_4)$, and $(B_4C_1C_3)$. Now $(\xi_2\xi_3\xi_4)$ transforms $B_4C_1C_3$ into $B_3C_1C_2$; $(\xi_2\xi_4\xi_3)$ transforms $B_2C_1C_4$ into $B_3C_1C_2$.

The commutative group $K_8$ contains the cyclic subgroups

$$C_4^5=(B_3C_1C_5), \qquad C_4^0=(B_3C_1C_0),$$

and a single further subgroup $G_4^3$ of order 4. But $C_2C_5$ transforms $C_4^0$ into $C_4^5$. Now $C_4^5$, whose substitutions of period 4 have the characteristic determinant $(1-\rho)(1+\rho)^2(1+\rho^2)$, is not conjugate with $C_4^3$, for which the corresponding quantity is $(1-\rho)(1+\rho^2)^2$.

The group $L_8$ contains a single cyclic group $C_4^5$ and but two further groups of order 4: $K_4^*$ and $K_4$. Now $B_2$ transforms $K_4$ into $K_4'$.

Finally, $T_8$ contains $C_4^0$, $K_4^*$, $K_4''''$, but no further groups of order 4.

THEOREM. *Within $O_\Omega$, every subgroup of order 4 is conjugate with one and but one of the six groups $G_4^2$, $K_4'$, $K_4^*$, $K_4''''$,*

$$C_4^3=(B_3C_1C_4), \qquad C_4^5=(B_3C_1C_5). \tag{48}$$

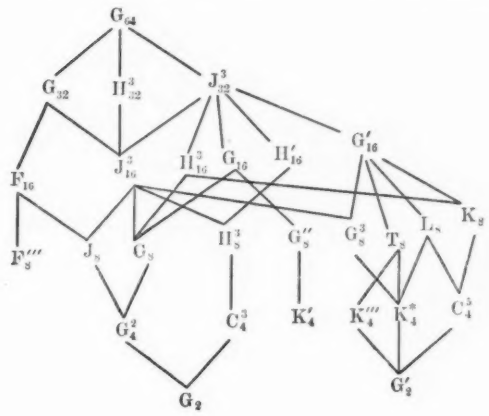
*The subgroups of order 2.*

20. There are exactly two distinct sets of conjugate operators of period 2 within the simple quaternary abelian group (*Linear Groups*, p. 105). The same consequently holds for $O_\Omega$. As representatives belonging to $G_{64}$, we may take $C_1C_2C_3C_4$ and $C_1C_3$, which generate the groups $G_2$ and $G_2'$, respectively.

THEOREM. *Within $O_\Omega$, every subgroup of order 2 is conjugate with $G_2$ or $G_2'$.*

*Summary of the subgroups of order a power of 2.*

21. Representatives of each distinct set of conjugate subgroups of order a power of 2 within the group $O_\Omega$, together with all their incidences, are exhibited in the following scheme:




LARGEST SUBGROUPS IN WHICH THE GROUPS OF ORDER A POWER OF 2 ARE SELF-CONJUGATE, §§ 22–47.

22. LEMMA I. *If, for* $p^n = 8l \pm 3$, *a substitution of* $O_\Omega$ *transforms* $C_0 C_t$ *into a substitution belonging to* $G_{960}$, *it replaces one of the variables by* $\pm \xi_t$.

Let $S$ have the matrix $(\alpha_{ij})$. Then $C_0 C_t$ replaces $\sum_{j=1}^{5} \alpha_{ij}\xi_j$ by

$$-\sum_{j=1,\cdots,5;\ j\neq t} \alpha_{ij}\xi_j + \alpha_{it}\xi_t = -\sum_{j=1}^{5} \alpha_{ij}\xi_j + 2\alpha_{it}\xi_t.$$

Since the matrix of $S^{-1}$ is $(\alpha_{ji})$, it follows that

$$S^{-1}C_0C_tS: \qquad \xi'_i = -\xi_i + 2\alpha_{it}\sum_{j=1}^{5}\alpha_{jt}\xi_j \qquad (i=1,\cdots,5).$$

Since 2 is a not-square, no one of the diagonal terms $-1+2\alpha_{it}^2$ of the latter is zero. But a substitution of $G_{960}$ has a single non-vanishing coefficient in each row (or column). Hence must

$$\alpha_{it}\alpha_{jt} = 0 \qquad (i,j=1,\cdots,5;\ j\neq i).$$

Hence the product of any two of the five coefficients in the $t$th column of the matrix of $S$ is zero, so that four are zero. It $\alpha_{rt}$ is the non-vanishing one, all the remaining coefficients in the $r$th row are zero in view of the orthogonal conditions. Hence $S$ replaces $\xi_r$ by $\alpha_{rt}\xi_t$, where $\alpha_{rt}^2 = 1$.

COROLLARY I. *If* $S$ *transforms each* $C_0C_t (t=1,2,3,4,5)$ *into a substitution of* $G_{960}$, *then* $S$ *itself belongs to* $G_{960}$.

COROLLARY II. *If* $S$ *transforms* $C_0C_t$ *into itself, it replaces* $\xi_t$ *by* $\pm\xi_t$:

Indeed, $-1+2\alpha_{it}^2 = -1$ gives $\alpha_{it}=0$ $(i=1,\cdots,5;\ i\neq t)$, whence, by the orthogonal conditions, $\alpha_{tj}=0 (j\neq t)$.

COROLLARY III. *If $S$ transforms into itself a subgroup of $G_{960}$ which contains a single $C_0C_t$, then $S$ replaces $\xi_t$ by $\pm\xi_t$.*

Indeed, $S$ transforms $C_0C_t$ into a substitution in whose matrix each diagonal term is $\neq 0$. Since the latter must belong to $G_{960}$, it is a product of the $C_i$. But $C_iC_j$ is not conjugate with $C_0C_t$, since they have distinct characteristic determinants. Hence $C_0C_t$ is transformed into itself.

23. LEMMA II. *If a quinary orthogonal substitution $S$ in any $GF[p^n]$, for which $p^n = 8l \pm 3$ or $8l-1$, transforms each $C_kC_t$ $(k, t = 1, 2, 3, 4)$ into a substitution replacing $\xi_5$ by $\pm\xi_5$, then $S$ replaces $\xi_5$ by one of the variables or its negative.*

Taking $(\alpha_{ij})$ as the matrix of $S$, we get for $S' = S^{-1}C_kC_tS$:

$$\xi_i' = \xi_i - 2\alpha_{ik}\sum_{j=1}^{5}\alpha_{jk}\xi_j - 2\alpha_{it}\sum_{j=1}^{5}\alpha_{jt}\xi_j \qquad (i=1,\cdots,5).$$

The conditions that $S'$ shall replace $\xi_5$ by $\pm\xi_5$ are

$$1 - 2\alpha_{5k}^2 - 2\alpha_{5t}^2 = \pm 1, \qquad \alpha_{5k}\alpha_{jk} + \alpha_{5t}\alpha_{jt} = 0 \quad (j=1,2,3,4).$$

According as the upper or lower sign holds, we have

$$\alpha_{5k}^2 + \alpha_{5t}^2 = 0 \qquad \text{or} \qquad \alpha_{5k}^2 + \alpha_{5t}^2 = 1.$$

In the first case, we have the five equations

$$\alpha_{5k}\alpha_{jk} + \alpha_{5t}\alpha_{jt} = 0 \qquad (j=1,\cdots,5).$$

But not all the determinants of the second order of the matrix formed of the $k$th and $t$th columns of $S$ are zero. Hence $\alpha_{5k} = \alpha_{5t} = 0$. If, in the second case, $\alpha_{5t} = 0$, then $\alpha_{5k} \neq 0$, $\alpha_{jk} = 0$ $(j = 1, 2, 3, 4)$, and $\xi_5' = \alpha_{5k}\xi_k$, in view of the orthogonal conditions.

Now, if every sum of two of the terms $\alpha_{51}^2$, $\alpha_{52}^2$, $\alpha_{53}^2$, $\alpha_{54}^2$ equals 1, each term equals $\frac{1}{2}$, whence $p^n = 8l \pm 1$. Then $2 + \alpha_{55}^2 = 1$, so that $p^n = 8l + 1$, contrary to assumption. Let next one such sum equal 0; for definiteness, $\alpha_{53}^2 + \alpha_{54}^2 = 0$. Then $\alpha_{53} = \alpha_{54} = 0$. Since $\alpha_{51}^2 + \alpha_{53}^2 = 0$ or 1, $\alpha_{51}^2 = 0$ or 1. Likewise, $\alpha_{52}^2 = 0$ or 1. But $\alpha_{51}^2 + \alpha_{52}^2 = 0$ or 1. Hence at least one of the terms $\alpha_{51}^2$, $\alpha_{52}^2$ vanishes. If both vanish, $\xi_5' = \alpha_{55}\xi_5$. If $\alpha_{52} \neq 0$, then $\alpha_{51} = 0$, and $\xi_5' = \alpha_{52}\xi_2$, as shown above.

COROLLARY. *If each transform leaves $\xi_5$ unaltered, $S$ replaces $\xi_5$ by $\pm\xi_5$.*

24. Since the $C_0C_t(t = 1, \cdots, 5)$ generate $G_{16}$, it follows from Corollary I to Lemma I that *a subgroup of $G_{960}$ containing $G_{16}$ is self-conjugate within $O_\Omega$ only under a subgroup of $G_{960}$.* Now the only even substitutions on $\xi_1, \cdots, \xi_5$ which transform $B_k$ $(k > 1)$ into itself are $B_1 = I, B_2, B_3, B_4$; while the only ones which transform $B_2, B_3, B_4$ amongst themselves are those of the alternating group on $\xi_1, \xi_2, \xi_3, \xi_4$.

THEOREM. *Within* $O_\Omega$, $G_{16}$ *is self-conjugate only under* $G_{960}$, $J_{32}^k$ *is self-conjugate only under* $G_{64}$, *while* $G_{64}$ *is self-conjugate only under*

$$(49)\quad G_{192} = \left\{ E_t,\ E_t C_i C_j \begin{pmatrix} i,j = 0,1,2,3,4,5;\ j>i \\ E_t \text{ ranging over even substitutions on } \xi_1, \cdots, \xi_4 \end{pmatrix} \right\}.$$

25. A substitution $S$ which is commutative with $C_1C_2C_3C_4$ replaces $\xi_5$ by $\pm\xi_5$ (Corollary II to Lemma I). By *Linear Groups*, p. 160, the number of quaternary orthogonal substitutions of determinant $+1$ is

$$(p^{3n}-p^n)(p^{2n}-1)p^n.$$

Exactly one half of these belong to $O_\Omega$; for, $S(\xi_1\xi_2)C_1$ is a quaternary orthogonal substitution of determinant $+1$ if $S$ is, while one and but one of the two belongs to $O_\Omega$. Hence the preceding number is the order of the subgroup of $O_\Omega$ commutative with $C_1C_2C_3C_4$. Another proof follows from the fact that $C_1C_2C_3C_4$ corresponds (*Linear Groups*, pp. 179–182) to the abelian substitution $T_{1,-1}$. The latter is commutative with exactly $[p^n(p^{2n}-1)]^2$ abelian operators.*

THEOREM. *Within* $O_\Omega$, $G_2$ *is self-conjugate only under* $G_{p^{2n}(p^{2n}-1)^2}$.

The last group can be given a very simple form when $p^n = 3$. Then

$$\alpha_{i1}^2+\alpha_{i2}^2+\alpha_{i3}^2+\alpha_{i4}^2 \equiv 1 \pmod 3 \qquad (i=1,2,3,4)$$

requires that one or four of the coefficients in each row of the matrix for $S$ shall $\neq 0$. In the former case, $S$ belongs to $G_{192}$. In the latter case, $CW^{\pm 1}$ replaces $\xi_1$ by $\sum_{j=1}^{j=4}\alpha_{1j}\xi_j$, $C$ being a suitably chosen product of an even number of the $C_i (i<5)$. Hence $S = CW^{\pm 1}\Gamma$, where $\Gamma$ leaves $\xi_1$ unaltered and replaces $\xi_5$ by $\pm\xi_5$, and therefore belongs to $G_{192}$. But $W$ transforms $C_1C_i$ into $B_iC_1C_2C_3C_4$, $C_1$ into $WC_1$, and $C_i$ into $WB_iC_iC_1C_2C_3C_4$ for $i=2,3,4$. Hence $S = W^{\pm 1}\Gamma_1$, where $\Gamma_1$ belongs to $G_{192}$. *Hence, for* $p^n = 3$, *the substitutions commutative with* $C_1C_2C_3C_4$ *form the group*

$$(50)\qquad G_{576} = \{\Gamma,\ W\Gamma,\ W^2\Gamma\ (\Gamma \text{ ranging over } G_{192})\}.$$

26. A substitution is commutative with $B_3C_1C_4$ if and only if it has the form $S'$ of § 11. The orthogonal conditions on $S'$ reduce to the four:

$$(51)\quad \begin{aligned} &\alpha_{11}^2+\alpha_{12}^2+\alpha_{13}^2+\alpha_{14}^2 = 1, && \alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}+\alpha_{13}\alpha_{23}+\alpha_{14}\alpha_{24} = 0,\\ &\alpha_{21}^2+\alpha_{22}^2+\alpha_{23}^2+\alpha_{24}^2 = 1, && -\alpha_{13}\alpha_{21}+\alpha_{14}\alpha_{22}+\alpha_{11}\alpha_{23}-\alpha_{12}\alpha_{24} = 0. \end{aligned}$$

* Transactions, vol. 2 (1901), bottom of p. 109. The number is the same for the quotient-group of order $\Omega$ since $P_{12}$ transforms $T_{1,-1}$ into $T_{2,-1} = T_{1,-1}\cdot T_{1,-1}\,T_{2,-1}$.

If $\alpha_{11}^2+\alpha_{13}^2\neq 0$, the equations (51) in the second column give

$$\alpha_{21}=r\alpha_{22}+s\alpha_{24},\qquad \alpha_{23}=s\alpha_{22}-r\alpha_{24}, \tag{52}$$

where

$$r=\frac{\alpha_{13}\alpha_{14}-\alpha_{11}\alpha_{12}}{\alpha_{11}^2+\alpha_{13}^2},\quad s=\frac{-\alpha_{11}\alpha_{14}-\alpha_{12}\alpha_{13}}{\alpha_{11}^2+\alpha_{13}^2},\quad r^2+s^2=\frac{\alpha_{12}^2+\alpha_{14}^2}{\alpha_{11}^2+\alpha_{13}^2}. \tag{53}$$

It follows that

$$\alpha_{21}^2+\alpha_{23}=(r^2+s^2)(\alpha_{22}^2+\alpha_{24}^2),\quad \sum_{j=1}^{4}\alpha_{2j}^2=\frac{(\alpha_{22}^2+\alpha_{24}^2)(\alpha_{11}^2+\alpha_{12}^2+\alpha_{13}^2+\alpha_{14}^2)}{\alpha_{11}^2+\alpha_{13}^2}.$$

The conditions (51) therefore reduce to (52) together with

$$\alpha_{11}^2+\alpha_{12}^2+\alpha_{13}^2+\alpha_{14}^2=1,\qquad \alpha_{22}^2+\alpha_{24}^2=\alpha_{11}^2+\alpha_{13}^2. \tag{54}$$

By *Linear Groups*, p. 46, the equation $\alpha_{11}^2+\alpha_{13}^2=\kappa$ has $p^n-\nu$ or $p^n+p^n\nu-\nu$ sets of solutions in the $GF[p^n]$, where $\nu=\pm 1$ according as $p^n=4l\pm 1$. Hence there are $p^{2n}-(2p^n+\nu p^n-2\nu)$ sets $\alpha_{11}$, $\alpha_{13}$ for which $\alpha_{11}^2+\alpha_{13}^2$ is neither 0 nor 1. Each such set furnishes $p^n-\nu$ sets $\alpha_{12}$, $\alpha_{14}$ satisfying $\alpha_{12}^2+\alpha_{14}^2=1-(\alpha_{11}^2+\alpha_{13}^2)$. Next, each of the $p^n-\nu$ sets of solutions of $\alpha_{11}^2+\alpha_{13}^2=1$ furnishes $p^n+p^n\nu-\nu$ sets $\alpha_{12}$, $\alpha_{14}$. Hence there are

$$(p^n-\nu)[(p^{2n}-2p^n-\nu p^n+2\nu)+(p^n+p^n\nu-\nu)]=(p^n-\nu)(p^{2n}-p^n+\nu)$$

sets $\alpha_{11},\cdots,\alpha_{14}$ satisfying the first condition * (54) and $\alpha_{11}^2+\alpha_{13}^2\neq 0$.

If $\alpha_{11}^2+\alpha_{13}^2=0$, then $\alpha_{12}^2+\alpha_{14}^2=1$. The last equations (51) now give

$$\alpha_{22}=\alpha\alpha_{21}+\beta\alpha_{23},\qquad \alpha_{24}=\beta\alpha_{21}-\alpha\alpha_{23}, \tag{52'}$$

where

$$\alpha=-\alpha_{11}\alpha_{12}+\alpha_{13}\alpha_{14},\qquad \beta=-\alpha_{12}\alpha_{13}-\alpha_{11}\alpha_{14}, \tag{53'}$$

$$\alpha^2+\beta^2=(\alpha_{11}^2+\alpha_{13}^2)(\alpha_{12}^2+\alpha_{14}^2)=0.$$

Hence $\alpha_{22}^2+\alpha_{24}^2=0$. The condition (51) therefore reduce to (52′) and

$$\alpha_{11}^2+\alpha_{13}^2=0,\qquad \alpha_{12}^2+\alpha_{14}^2=1,\qquad \alpha_{21}^2+\alpha_{23}^2=1. \tag{54'}$$

and hence have $(p^n-\nu)^2(p^n+p^n\nu-\nu)$ sets of solutions $\alpha_{ij}$.

The total number of sets of solutions of (51) is thus $(p^n-\nu)^2(p^{2n}+p^n\nu)$.

The determinant of $S'$ is seen to equal

$$\pm\{(\alpha_{11}^2+\alpha_{12}^2+\alpha_{13}^2+\alpha_{14}^2)(\alpha_{21}^2+\alpha_{22}^2+\alpha_{23}^2+\alpha_{24}^2)$$

$$-(\alpha_{11}\alpha_{21}+\alpha_{12}\alpha_{22}+\alpha_{13}\alpha_{23}+\alpha_{14}\alpha_{24})^2-(-\alpha_{13}\alpha_{21}+\alpha_{14}\alpha_{22}+\alpha_{11}\alpha_{23}-\alpha_{12}\alpha_{24})^2\}$$

and hence by (51) equals $\pm 1$. The sign $\pm$ must therefore be taken $+$.

* Since this has $p^{3n}-p^n$ sets of solutions (*Linear Groups*, p. 47), we obtain a second proof.

For $p^n = 3$, only half of the resulting 96 orthogonal substitutions $S'$ of determinant $+1$ belong to $O_\Omega$. These are seen to be

$$(55)\quad B_i,\ B_iC_1C_3,\ B_iC_2C_4,\ B_iC_1C_2C_3C_4,\ B_jC_1C_2,\ B_jC_2C_3,\ B_jC_1C_4,\ B_jC_3C_4 \qquad (i=1,4;\ j=2,3)$$

together with their products on the left by $W(\xi_2\xi_4\xi_3)$ and its inverse $W^2(\xi_2\xi_3\xi_4)$.

For $p^n = 5$, it will be shown that exactly half of the resulting 480 orthogonal substitutions $S'$ of determinant $+1$ belong to $O_\Omega$. Assuming first that 3 of the $\alpha_{1j}$ are zero, we obtain the 16 substitutions (55) and 16 others not in $O_\Omega$. Assume next that exactly one of the $\alpha_{1j}$ is zero. Then two of the $\alpha_{1j}^2$ are $+1$ and one is $-1$, so that there are 12 types. For example,* take $\alpha_{11}^2 = \alpha_{12}^2 = +1$, $\alpha_{13}^2 = -1$, $\alpha_{14} = 0$. By (54′), $\alpha_{21}^2 + \alpha_{23}^2 = 1$. Hence either $\alpha_{21} = 0$, whence $\alpha_{22} = -\alpha_{12}\alpha_{13}\alpha_{23}$, $\alpha_{24} = \alpha_{11}\alpha_{12}\alpha_{23}$ by (52′), or else $\alpha_{23} = 0$, whence

$$\alpha_{22} = -\alpha_{11}\alpha_{12}\alpha_{21}, \qquad \alpha_{24} = -\alpha_{12}\alpha_{13}\alpha_{21}.$$

In the respective cases, $S'$ becomes

$$S_1' = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{12}\alpha_{13}\alpha_{22} & \alpha_{11}\alpha_{13}\alpha_{22} & 0 \\ -\alpha_{13} & 0 & \alpha_{11} & -\alpha_{12} & 0 \\ \alpha_{12}\alpha_{13}\alpha_{22} & -\alpha_{11}\alpha_{13}\alpha_{22} & 0 & \alpha_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \alpha_{11}^2 = \alpha_{12}^2 = +1 \\ \alpha_{13}^2 = \alpha_{22}^2 = -1 \end{pmatrix}$$

$$S_2' = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ -\alpha_{11}\alpha_{12}\alpha_{22} & \alpha_{22} & 0 & \alpha_{11}\alpha_{13}\alpha_{22} & 0 \\ -\alpha_{13} & 0 & \alpha_{11} & -\alpha_{12} & 0 \\ 0 & -\alpha_{11}\alpha_{13}\alpha_{22} & \alpha_{11}\alpha_{12}\alpha_{22} & \alpha_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \alpha_{11}^2 = \alpha_{12}^2 = \alpha_{22}^2 = +1 \\ \alpha_{13}^2 = -1 \end{pmatrix}.$$

To show that none of the 16 substitutions $S_1'$ belong to $O_\Omega$, denote $S_1'$ by $S_1^*$ when $\alpha_{11} = \alpha_{12} = +1$, $\alpha_{13} = +2$. According as $\alpha_{22} = +2$ or $-2$, we have for $S_1^*$

$$R_{123}C_3R_{234}(\xi_2\xi_4)C_2C_3, \qquad \text{or} \qquad R_{123}C_3R_{234}(\xi_2\xi_4)C_3C_4,$$

neither of which belongs to $O$. Giving to $(\alpha_{11}, \alpha_{12}, \alpha_{13})$ in turn the values $(1,1,-2), (-1,-1,2), (-1,-1,-2), (1,-1,2), (1,-1,-2),$

* Note that one of the four $S'$, $B_4S'$, $B_3C_1C_2S'$, $B_2C_1C_2S'$ has $\alpha_{14}=0$, while each is commutative with $B_3C_1C_4$. Also, $(\xi_1\xi_2\xi_3)S'$ has $\alpha_{11}^2=-1$, $\alpha_{12}^2=\alpha_{13}^2=+1$, and belongs to $O_\Omega$ if and only if $S'$ does.

$-1, 1, 2)$, $(-1, 1, -2)$, we find for $S_1'$: $C_3C_4S_1^*C_3C_4$, $C_3C_4S_1^*C_1C_4$, $S_1^*C_1C_3$, $C_1C_3S_1^*C_1C_3$, $C_1C_4S_1^*C_1C_4$, $C_1C_4S_1^*C_3C_4$, $C_1C_3S_1^*$.

To show that all the 16 substitutions $S_2'$ belong to $O_\Omega$, denote $S_2'$ by $S_2^*$ when $\alpha_{11}=\alpha_{12}=+1$, $\alpha_{13}=+2$. According as $\alpha_{22}=+2$ or $-2$, we have for $S_2^*$

$$R_{123}C_2C_5R_{234}C_2C_5, \qquad \text{or} \qquad R_{123}C_3C_4R_{234}C_1C_4.$$

Giving to $(\alpha_{11}, \alpha_{12}, \alpha_{13})$ in turn the values $(1, 1, -2)$, $(-1, -1, 2)$, $-1, -1, -2)$, $(1, -1, 2)$, $(1, -1, -2)$, $(-1, 1, 2)$, $(-1, 1, -2)$, we find for $S_2'$: $C_3C_4S_2^*C_3C_4$, $C_3C_4S_2^*C_1C_4$, $S_2^*C_1C_3$, $C_1C_3S_2C_1C_3$, $C_1C_4S_2C_1C_4$, $C_1C_4S_2^*C_3C_4$, $C_1C_3S_2^*$.

Assume lastly that none of the $\alpha_{1j}$ are zero. Then every $\alpha_{1j}^2\equiv -1$. By (54), $\alpha_{22}^2+\alpha_{24}^2\equiv -2$, so that $\alpha_{21}^2+\alpha_{23}^2\equiv -2 \pmod 5$. Hence every $\alpha_{2j}^2\equiv -1$. By (53),

$$r=2(\alpha_{13}\alpha_{14}-\alpha_{11}\alpha_{12}),\ s=3(\alpha_{11}\alpha_{14}+\alpha_{12}\alpha_{13}),\ rs\equiv 0.$$

Let first $s\equiv 0$, so that $\alpha_{14}\equiv \alpha_{11}\alpha_{12}\alpha_{13}$, $r\equiv \alpha_{11}\alpha_{12}$. By (52) we find for $S'$:

$$S_3'=\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{11}\alpha_{12}\alpha_{13} & 0 \\ \alpha_{11}\alpha_{12}\alpha_{22} & \alpha_{22} & -\alpha_{11}\alpha_{12}\alpha_{24} & \alpha_{24} & 0 \\ -\alpha_{13} & \alpha_{11}\alpha_{12}\alpha_{13} & \alpha_{11} & -\alpha_{12} & 0 \\ -\alpha_{11}\alpha_{12}\alpha_{24} & -\alpha_{24} & -\alpha_{11}\alpha_{12}\alpha_{22} & \alpha_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Denote $S_3'$ by $S_3^*$ when $\alpha_{11}=\alpha_{12}=\alpha_{13}=+2$. For $\alpha_{22}=\alpha_{24}=+2$, we have for $S_3^*$

$$S_3^{**}=C_2C_4(\xi_2\xi_4\xi_3)R_{234}C_2C_4R_{124}(\xi_1\xi_3\xi_2)R_{123}(\xi_1\xi_4\xi_2)C_3C_4.$$

For $\alpha_{22}=\alpha_{24}=-2$, $S_3^*=S_3^{**}C_2C_4$. For $\alpha_{22}=2$, $\alpha_{24}=-2$, $S_3^*$ becomes

$$S_3^{***}=C_2C_4(\xi_2\xi_4\xi_3)R_{234}C_2C_4R_{124}(\xi_1\xi_2\xi_3)R_{123}(\xi_1\xi_4\xi_3\xi_2)C_3.$$

For $\alpha_{22}=-2$, $\alpha_{24}=+2$, $S_3^*$ becomes $S_3^{***}C_2C_4$. Hence $S_3^*$ belongs to $O_\Omega$ if and only if $\alpha_{22}=\alpha_{24}$. Next, $S_3'$ becomes $C_2C_4C_3^*C_2C_4$ when $\alpha_{11}=\alpha_{13}=2$, $\alpha_{12}=-2$; hence must $\alpha_{22}=\alpha_{24}$. Again, $S_3'$ becomes $C_2C_4S_3^*C_1C_2C_3C_4$ when $\alpha_{11}=\alpha_{13}=-2$, $\alpha_{12}=+2$, whence must $\alpha_{22}=\alpha_{24}$. Also, $S_3'=S^*C_1C_3$ when $\alpha_{11}=\alpha_{22}=-2$, $\alpha_{13}=-2$, whence must $\alpha_{22}=\alpha_{24}$. Denote $S_3'$ by $[\alpha, \beta]$ when $\alpha_{11}=\alpha_{12}=2$, $\alpha_{13}=-2$. Then $[\alpha, -\beta]=C_3C_4S_3^*C_3C_4$, whence must $\alpha_{22}=-\alpha_{24}$. For $\alpha_{11}=\alpha_{12}=-2$, $\alpha_{13}=+2$, $S_3'=[\alpha, \beta]C_1C_3$, whence must $\alpha_{22}=-\alpha_{24}$. For $\alpha_{11}=2$, $\alpha_{12}=\alpha_{13}=-2$, $S_3'=C_2C_4[-\alpha, -\beta]$, whence must $\alpha_{22}=-\alpha_{24}$. Finally, $S_3'=C_2C_4[-\alpha, -\beta]C_1C_3$, when $\alpha_{11}=-2$, $\alpha_{12}=\alpha_{13}=+2$, whence must $\alpha_{22}=-\alpha_{24}$. To summarize, $S_3'$ belongs to $O_\Omega$ only when $\alpha_{22}=+\alpha_{24}$ if $\alpha_{11}=+\alpha_{13}$, and $\alpha_{22}=-\alpha_{24}$ if $\alpha_{11}=-\alpha_{13}$, or briefly, only when $\alpha_{24}=-\alpha_{11}\alpha_{13}\alpha_{22}$.

Let next $r \equiv 0$, so that $\alpha_{14} \equiv -\alpha_{11}\alpha_{12}\alpha_{13}$, $s \equiv \alpha_{12}\alpha_{13}$. Then, by (52),

$$\alpha_{21} = \alpha_{12}\alpha_{13}\alpha_{24}, \qquad \alpha_{23} = \alpha_{12}\alpha_{13}\alpha_{22}.$$

For $\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{22} = \alpha_{24} = +2$, $S'$ becomes * $\Sigma$ of § 11 and hence belongs to $O_\Omega$. Hence, in view of the preceding case, the general $S'$, with $r = 0$, belongs to $O_\Omega$ only when $\alpha_{24} = -\alpha_{11}\alpha_{13}\alpha_{22}$.

We may combine the two preceding cases as follows: An orthogonal substitution $S'$ with every $\alpha_{ij} \neq 0$ belongs to $O_\Omega$ if and only if

$$\alpha_{21} = \alpha_{11}\alpha_{12}\alpha_{22}, \qquad \alpha_{23} = \alpha_{11}\alpha_{14}\alpha_{22}, \qquad \alpha_{24} = -\alpha_{11}\alpha_{13}\alpha_{22}. \tag{56}$$

*Hence of the* 480 *orthogonal substitutions of determinant unity which are commutative with* $B_3C_1C_4$, *exactly* 240 *belong to* $O_\Omega$ *for* $p^n = 5$.

In the general case there are exactly $\frac{1}{2}(p^n - \nu)(p^{2n} - 1)p^n$ substitutions of $O_\Omega$ commutative with $B_3C_1C_4$, where $\nu = \pm 1$ according as $p^n = 4l \pm 1$. Indeed, $S_1 = (\xi_1\xi_3)C_1S$ is commutative with $B_3C_1C_4$ if $S$ is, while only one of the pair $S$, $S_1$ belongs to $O_\Omega$ by §§ 3, 4.

Now $B_3$ transforms $B_3C_1C_4$ into its inverse $B_3C_2C_3$.

THEOREM. *Within* $O_\Omega$, *the group* $C_4^3 = (B_3C_1C_4)$ *is self-conjugate only under a group* $G_{(p^n-\nu)(p^{2n}-1)p^n}$.

27. We may now readily determine the largest subgroup transforming $G_{32}$ into itself. The latter has exactly 12 substitutions of period 4: $B_kC_lC_l$, $k, l = 2, 3, 4$; $k \neq l$. They are all conjugate within $G_{192}$, under which $G_{32}$ is certainly self-conjugate. Indeed, $B_3$ and $C_2C_5$ transform $B_3C_1C_4$ into $B_3C_2C_3$ and $B_3C_1C_2$, respectively: $(\xi_2\xi_3\xi_4)$ and $(\xi_2\xi_4\xi_3)$ transform $B_3C_1C_2$ into $B_2C_1C_4$ and $B_4C_1C_3$, respectively: $C_2C_5$ transforms $B_4C_1C_3$ into $B_4C_1C_2$; $B_3$ transforms $B_3C_1C_2$ into $B_3C_3C_4$, $B_4C_1C_2$ into $B_4C_3C_4$, and $B_2C_1C_4$ into $B_2C_2C_3$; $C_2C_5$ transforms $B_2C_2C_3$ into $B_2C_1C_3$; $B_2$ transforms $B_2C_1C_3$ into $B_2C_2C_4$, and $B_4C_1C_3$ into $B_4C_2C_4$.

We next show that exactly 48 operators of $O_\Omega$ transform $G_{32}$ and the substitution $B_3C_1C_4$ each into itself. It will then follow that $G_{32}$ is self-conjugate only under a group of order $12 \times 48$.

For $p^n = 3$, this result follows from § 26 since $W^2(\xi_2\xi_3\xi_4)$ transforms $G_{32}$ into itself (§ 11).

For $p^n = 5$ consider in turn the various types of substitutions of $O_\Omega$ which are commutative with $B_3C_1C_4$. When 3 of the $\alpha_{ij}$ are zero, there resulted the 16 substitutions (55). Since they belong to $G_{192}$, they transform $G_{32}$ into itself. When a single $\alpha_{ij}$ is zero, there resulted 12 types of substitutions, one type comprising the 16 substitutions $S_2'$, the substitutions of the remaining types being of the form $\Gamma S_2'$, where $\Gamma$ belongs to $G_{192}$. But $S_2'$ transforms $C_1C_4$ into

* Note that $\Sigma = C_4 S_3'' C_4(\xi_2\xi_4\xi_3)$.

$$\begin{pmatrix} -1 & 2\alpha_{12}\alpha_{22} & 2\alpha_{11}\alpha_{13} & 0 & 0 \\ 2\alpha_{12}\alpha_{22} & 1 & 0 & 3\alpha_{11}\alpha_{13} & 0 \\ 2\alpha_{11}\alpha_{13} & 0 & 1 & 2\alpha_{12}\alpha_{22} & 0 \\ 0 & 3\alpha_{11}\alpha_{13} & 2\alpha_{12}\alpha_{22} & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \alpha_{11}^2=\alpha_{12}^2=\alpha_{22}^2=1 \\ \alpha_{13}^2=-1 \end{pmatrix},$$

which does not belong to $G_{32}$ since its non-diagonal terms do not all vanish. Hence the 12 types are all excluded. Finally, when none of the $\alpha_{ij}$ are zero, there resulted the 32 substitutions $S$ of the form $S'$ with every $\alpha_{1j}^2=\alpha_{2j}^2=-1$ and satisfying (56). We verify that $S$ transforms $C_1C_4$ into

$$(57) \qquad \begin{pmatrix} 0 & \lambda & \mu & 0 & 0 \\ \lambda & 0 & 0 & -\mu & 0 \\ \mu & 0 & 0 & \lambda & 0 \\ 0 & -\mu & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \lambda=2\alpha_{12}\alpha_{22}+2\alpha_{11}\alpha_{13}\alpha_{14}\alpha_{22} \\ \mu=2\alpha_{11}\alpha_{13}+2\alpha_{12}\alpha_{14} \end{pmatrix}.$$

Since $\lambda^2+\mu^2\equiv 1$, either $\lambda=0$ or $\mu=0$. If $\lambda=0$, then $\alpha_{14}=-\alpha_{11}\alpha_{12}\alpha_{13}$, $\mu=-\alpha_{11}\alpha_{13}$, and (57) is $B_3C_1C_3$ or $B_3C_2C_4$. If $\mu=0$, then $\alpha_{14}=\alpha_{11}\alpha_{12}\alpha_{13}$, $\lambda=-\alpha_{11}\alpha_{22}$, and (57) is either $B_2$ or $B_2C_1C_2C_3C_4$. Hence (57) belongs to $G_{32}$ in every case.

Next, $S$ transforms $C_1C_2$ into

$$(58) \qquad \begin{pmatrix} 0 & 0 & \sigma & \rho & 0 \\ 0 & 0 & -\rho & \sigma & 0 \\ \sigma & -\rho & 0 & 0 & 0 \\ \rho & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \sigma=2\alpha_{11}\alpha_{13}-2\alpha_{12}\alpha_{14} \\ \rho=2\alpha_{14}\alpha_{22}-2\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{22} \end{pmatrix}.$$

Since $\rho^2+\sigma^2\equiv 1$, either $\rho=0$ or $\sigma=0$. If $\rho=0$, then $\alpha_{14}=\alpha_{11}\alpha_{12}\alpha_{13}$ and (58) is either $B_3$ or $B_3C_1C_2C_3C_4$. If $\sigma=0$, then $\alpha_{14}=-\alpha_{11}\alpha_{12}\alpha_{13}$ and (58) is either $B_4C_1C_4$ or $B_4C_2C_3$. Hence (58) belongs to $G_{32}$ in every case.

Finally, $S$ transforms $B_2$ into

$$(59) \qquad \begin{pmatrix} \alpha & 0 & \beta & 0 & 0 \\ 0 & -\alpha & 0 & -\beta & 0 \\ \beta & 0 & -\alpha & 0 & 0 \\ 0 & -\beta & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \alpha=2\alpha_{11}\alpha_{12}+2\alpha_{13}\alpha_{14} \\ \beta=2\alpha_{11}\alpha_{14}-2\alpha_{12}\alpha_{13} \end{pmatrix}.$$

Then $\alpha^2+\beta^2\equiv 1$. If $\alpha=0$, then $\alpha_{14}=\alpha_{11}\alpha_{12}\alpha_{13}$ and (59) is either $B_3C_1C_3$ or $B_3C_2C_4$. If $\beta=0$, then $\alpha_{14}=-\alpha_{11}\alpha_{12}\alpha_{13}$ and (59) is either $C_1C_4$ or $C_2C_3$. Hence (59) belongs to $G_{32}$ in every case.

The general case will be established indirectly. Of the substitutions transforming $B_3C_1C_4$ into itself and hence also its inverse $B_3C_2C_3$ into itself, $B_4$ transforms $B_3C_1C_2$ into $B_3C_3C_4$; $\Sigma$ of § 11 transforms $B_3C_1C_2$ into $B_4C_1C_3$, and the latter into $B_2C_1C_4$; $B_4$ transforms $B_4C_1C_3$ into $B_4C_2C_4$; $B_2C_1C_2$ transforms $B_2C_1C_4$ into $B_2C_2C_3$. Hence 6 of the 12 substitutions of period 4 in $G_{32}$ are conjugate with $B_3C_1C_2$ by means of substitutions transforming $G_{32}$ and $B_3C_1C_4$ each into itself. We next show that no substitution of $O_\Omega$ transforms $B_3C_1C_4$ into itself and $B_3C_1C_2$ into one of the four: $B_4C_1C_2$, $B_4C_3C_4$, $B_2C_1C_3$, $B_2C_2C_4$. The condition $B_3C_1C_2S'=S'B_4C_1C_2$, where $S'$ is given in § 11, requires that every $\alpha_{1j}=\alpha_{2j}=0$, and hence is impossible. Likewise, $B_3C_1C_2S'=S'B_2C_1C_3$ is impossible. But $B_4$ transforms $B_4C_1C_2$ into $B_4C_3C_4$, and $B_2C_1C_3$ into $B_2C_2C_4$. Finally, we show that exactly 8 substitutions of $O_\Omega$ transform $B_3C_1C_4$ and $B_3C_1C_2$ each into itself. It suffices to find the substitutions which are commutative with both $B_3C_1C_4$ and $C_2C_4$. Now $C_2C_4S'=S'C_2C_4$ requires that $\alpha_{12}$, $\alpha_{14}$, $\alpha_{21}$, $\alpha_{23}$ all vanish. The resulting special form $S''$ of $S'$ transforms $C_1C_4$ into

$$(60)\qquad \begin{Vmatrix} \alpha_{13}^2-\alpha_{11}^2 & 0 & 2\alpha_{11}\alpha_{13} & 0 & 0 \\ 0 & \alpha_{22}^2-\alpha_{24}^2 & 0 & -2\alpha_{22}\alpha_{24} & 0 \\ 2\alpha_{11}\alpha_{13} & 0 & \alpha_{11}^2-\alpha_{13}^2 & 0 & 0 \\ 0 & -2\alpha_{22}\alpha_{24} & 0 & \alpha_{24}^2-\alpha_{22}^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

This belongs to $G_{32}$, when 2 is a not-square, if and only if $\alpha_{11}\alpha_{13}=0$, $\alpha_{22}\alpha_{24}=0$, since the only conditions on $S''$ are $\alpha_{11}^2+\alpha_{13}^2=1$, $\alpha_{22}^2+\alpha_{24}^2=1$. For $\alpha_{11}=0$, $S''$ belongs to $O_\Omega$ if and only if $\alpha_{22}=0$, whence $S''$ is $B_3C_1C_2$, $B_3C_1C_4$, $B_3C_2C_3$ or $B_3C_3C_4$, all belonging to $G_{32}$. For $\alpha_{13}=0$, then $\alpha_{24}=0$, whence $S''$ is $I$, $C_1C_3$, $C_2C_4$, or $C_1C_2C_3C_4$, all belonging to $G_{32}$.

THEOREM. *Within $O_\Omega$, the group $G_{32}$ is self-conjugate only under*

$$(61)\qquad G_{576}=\{\Gamma,\ \Sigma\Gamma,\ \Sigma^2\Gamma\ (\Gamma \textit{ ranging over } G_{192})\}.$$

28. The group $H_{32}^3$ is self-conjugate under $G_{64}$ by § 7. Of the 20 substitutions of period 4 in $H_{32}^3$, the four $B_3C_1C_2$, $B_3C_1C_4$, $B_3C_2C_3$, $B_3C_3C_4$ are conjugate within $G_{64}$; likewise the eight $B_2C_iC_5$, $B_2C_iC_0$ $(i=1,2,3,4)$; likewise the eight $B_4C_iC_5$, $B_4C_iC_0$, as follows from the table of conjugate substitutions of $G_{64}$ (§ 6). Now $B_2C_1C_5$ and $B_3C_1C_4$ have the characteristic determinants $(1-\rho)(1+\rho)^2(1+\rho^2)$ and $(1-\rho)(1+\rho^2)^2$, respectively (end of

§ 19). Hence $B_3C_1C_4$ is conjugate with only 4 of the substitutions of period 4 of $H_{32}^3$. We proceed to show that only 16 substitutions of $O_\Omega$ transform $H_{32}^3$ and $B_3C_1C_4$ each into itself and that the 16 are the substitutions (55) belonging to $G_{64}$. The proof is similar to that in § 27. Consider first the case $p^n = 5$. Then (57) belongs to $H_{32}^3$ if and only if $\alpha_{14} = -\alpha_{11}\alpha_{12}\alpha_{13}$; (58) belongs to $H_{32}^3$ if and only if $\alpha_{14} = +\alpha_{11}\alpha_{12}\alpha_{13}$. Hence a transformer with each $\alpha_{1j} \neq 0$ is excluded. Those with a single $\alpha_{1j}$ equal zero are excluded as in § 27. For the general case we proceed as at the end of § 27. The only substitutions transforming $B_3C_1C_4$ and $B_3C_1C_2$ each into itself are 8 substitutions belonging to $H_{32}^3$. Indeed, (60) belongs to $H_{32}^3$, when 2 is a not-square, if and only if $\alpha_{11}\alpha_{13} = 0$, $\alpha_{22}\alpha_{24} = 0$.

THEOREM. *Within $O_\Omega$, the group $H_{32}^3$ is self-conjugate only under $G_{64}$.*

29. The group $J_{16}^3$ is self-conjugate under $G_{64}$ since it is self-conjugate under both $G_{32}$ and $J_{32}^3$ (§§ 8, 10). Within $G_{64}$ the four substitutions of period 4 of $J_{16}^3$ are conjugate with $B_3C_1C_4$. It therefore remains only to determine all the substitutions $S$ of $O_\Omega$ which transform $J_{16}^3$ and $B_3C_1C_4$ each into itself. We proceed as in § 27. For $p^n = 5$, the only substitutions $S$ are the 16 substitutions (55); for, (57) belongs to $J_{16}^3$ if and only if $\alpha_{14} = -\alpha_{11}\alpha_{12}\alpha_{13}$, while (58) belongs $J_{16}^3$ if and only if $\alpha_{14} = +\alpha_{11}\alpha_{12}\alpha_{13}$.

In the general case, $S'$ belongs to $G_{64}$ if it is commutative with $C_2C_4$ (end of § 27). Within $G_{64}$ the substitutions of period 2 in $J_{16}^3$ fall into sets of conjugates as follows:

$$C_2C_4,\ C_1C_3;\ C_1C_2,\ C_3C_4;\ C_1C_4,\ C_2C_3;\ B_3,\ B_3C_1C_3,\ B_3C_2C_4,\ B_3C_1C_2C_3C_4.$$

The conditions for $C_2C_4S' = S'C_1C_2$ are $\alpha_{1j} = \alpha_{2j} = 0\,(j = 1, 2, 3, 4)$. Likewise, $S'$ cannot transform $C_2C_4$ into $C_1C_4$, nor into $B_3$.

THEOREM. *Within $O_\Omega$, the group $J_{16}^3$ is self-conjugate only under $G_{64}$.*

30. Since $G_{16}'$ contains $C_1C_0$, $C_3C_0$ and $C_5C_0$, a substitution $S$ commutative with it must replace three variables by $\pm\xi_1$, $\pm\xi_3$, $\pm\xi_5$ (Lemma I, § 22). Since further there exists an even substitution on $\xi_1, \cdots, \xi_5$ which replaces $\xi_1, \xi_3, \xi_5$ by those three variables, respectively, we may set $S = O_{2,4}^{\lambda,\mu}\Gamma$, where $\Gamma$ belongs to $G_{960}$. Now $O_{2,4}^{\lambda,\mu}$ transforms $B_3$ into $T \equiv (\xi_1\xi_3)T_1$, where $T_1$ replaces $\xi_2$ by $2\lambda\mu\xi_2 + (\lambda^2 - \mu^2)\xi_4$. In order that $T$ shall belong to $G_{16}'$, it is necessary that $T_1 = (\xi_2\xi_4)C$, where $C$ is a product of the $C_i$. Hence $\lambda\mu = 0$. The case $\lambda = 0$ is excluded if $S$ belongs to $O_\Omega$. Hence $O_{2,4}^{\lambda,\mu} = I$ or $C_2C_4$. Hence $S$ belongs to $G_{960}$. But the only even substitutions on $\xi_1, \cdots, \xi_5$ which transform $B_3$ into itself are $I, B_2, B_3, B_4$. But neither $B_2$ nor $B_4$ transforms $C_1C_0$, $C_3C_0$, $C_5C_0$ amongst themselves.

THEOREM. *Within $O_\Omega$, the group $G_{16}'$ is self-conjugate only under $J_{32}^3$.*

31. By a proof entirely analogous to the preceding, we obtain the

THEOREM. *Within $O_\Omega$, the group $H_{16}'$ is self-conjugate only under $J_{32}^3$.*

32. A substitution $S$ which transforms $G_8^1$ into itself must replace $\xi_5$ by $\pm \xi_5$ (Corollary III of § 22). If $S$ transforms $C_1C_3$ of $G_8^1$ into itself, then $S = O_{1,3}O_{2,4}C$, where $C$ is a product of $C_i$. Now $O_{1,3}^{\lambda,\mu}O_{2,4}^{\rho,\sigma}$ transforms $B_3$ into a substitution $B'$ which replaces $\xi_1$ and $\xi_2$ by

$$2\lambda\mu\xi_1 + (\lambda^2 - \mu^2)\xi_3, \qquad 2\rho\sigma\xi_2 + (\rho^2 - \sigma^2)\xi_4,$$

respectively. Since $\lambda^2 + \mu^2 = 1$ and 2 is a not-square, then $\lambda^2 - \mu^2 \neq 0$. Hence $\lambda\mu = 0$, $\rho\sigma = 0$ if $B'$ belongs to $G_8^1$, so that $S$ belongs to $(G_{16}, B_3)$. Now $G_8^3$ is evidently self-conjugate under $G_{64}$. Within the latter, $C_1C_3$ and $C_2C_4$ are conjugate, as also $B_3$, $B_3C_1C_3$, $B_3C_2C_4$, $B_3B_1C_2C_3C_4$. Hence if $O_\Omega$ contains a substitution which transforms $C_1C_3$ into $B_3$ and $G_8^3$ into itself, $G_8^3$ will be self-conjugate under exactly $6 \times 32$ substitutions of $O_\Omega$. Now an orthogonal substitution of period 2 replaces $\xi_5$ by $\pm \xi_5$ and transforms $C_1C_3$ into $B_3$ if and only if it has the form

$$\left\{\begin{array}{ccccc} \alpha_{11} & \alpha_{12} & -\alpha_{11} & -\alpha_{12} & 0 \\ \alpha_{12} & \alpha_{22} & \alpha_{12} & \alpha_{22} & 0 \\ -\alpha_{11} & \alpha_{12} & \alpha_{11} & -\alpha_{12} & 0 \\ -\alpha_{12} & \alpha_{22} & -\alpha_{12} & \alpha_{22} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{array}\right\} \qquad \left\{\begin{array}{l} 4\alpha_{11}^2 = 1 \\ 4\alpha_{12}^2 = 1 \\ 4\alpha_{22}^2 = 1 \end{array}\right\}.$$

It therefore transforms $B_3$ into $C_1C_3$ and $B_3C_1C_3$ and $B_3C_2C_4$ into themselves, and hence $G_8^1$ into itself. We choose the sign $\pm$ to make the determinant equal $+1$. If $S$ is one such substitution, then $S_1 = S(\xi_1\xi_3)C_5$ is another, since $(\xi_1\xi_3)C_5$ transforms each substitution of $G_8^3$ into itself. But * either $S$ or $S_1$ belongs to $O_\Omega$ (§ 4).

THEOREM. *Within $O_\Omega$, the group $G_8^3$ is self-conjugate only under $H_{192}$.*

33. Since $G_8''$ contains $C_2C_0$, $C_4C_0$ and $C_5C_0$, a substitution commutative with $G_8''$ has (as in § 30) the form $O_{1,3}^{\lambda,\mu}\Gamma$, $\Gamma$ in $G_{960}$. The first factor is evidently commutative with every substitution of $G_8''$. It belongs to $O_\Omega$ if and only if it is a $Q_{1,3}$ (of § 3), the number of which is $\frac{1}{2}(p^n - \nu)$. But the only even substitutions on $\xi_1, \cdots, \xi_5$ which transforms $C_2C_0$, $C_4C_0$ and $C_5C_0$ amongst themselves are

$$(62) \quad I,\ (\xi_1\xi_3)(\xi_2\xi_4),\ (\xi_1\xi_3)(\xi_2\xi_5),\ (\xi_1\xi_3)(\xi_4\xi_5),\ (\xi_2\xi_4\xi_5),\ (\xi_2\xi_5\xi_4).$$

THEOREM. *Within $O_\Omega$, the group $G_8''$ is self-conjugate only under*

$$(63) \qquad H_{2(p^n-\nu)} = [\,Q_{1,3}^{\lambda,\mu},\ G_{16}, (62)\,].$$

For $p^n = 3$ or 5, the only $Q_{1,3}^{\lambda,\mu}$ are $I$ and $C_1C_3$. Hence $H_{96} = [G_{16}, (62)]$.

* For $p^n = 5$, the values $\alpha_{11} = \alpha_{12} = \alpha_{22} = 2$, $\pm 1 = -1$, make the transformer equal to $C_2C_3C_4C_5(\xi_2\xi_4\xi_3)R_{234}C_2C_4R_{124}(\xi_1\xi_4\xi_3)R_{234}$.

34. The group $G_8$ is evidently self-conjugate under $G_{192}$ of § 24. Within the latter $C_1C_3$ is conjugate with $C_1C_2$, $C_1C_4$, $C_2C_3$, $C_2C_4$ and $C_3C_4$. It thus remains to determine the substitutions $S$ which are commutative with both $C_1C_3$ and $G_8$. As in § 32, $S = O_{1,3}O_{2,4}C$. But $O_{1,3}^{\lambda,\mu}$ transforms $C_1C_2$ into a substitution which replaces $\xi_1$ and $\xi_2$ by

$$(\mu^2 - \lambda^2)\xi_1 + 2\lambda\mu\xi_3, \qquad (\sigma^2 - \rho^2)\xi_2 + 2\rho\sigma\xi_4,$$

respectively. Hence must $\lambda\mu = 0$, $\rho\sigma = 0$.

THEOREM. *Within $O_\Omega$, the group $G_8$ is self-conjugate only under $G_{192}$.*

35. A substitution $S$ commutative with $K_8$ must replace $\xi_5$ by $\pm\xi_5$ (Corollary II of § 22), and must transform $C_1C_3$ into itself or $C_2C_4$. Hence $S = O_{1,3}O_{2,4}C$ or its product on the right by $B_2$. Now $O_{1,3}^{\lambda,\mu}O_{2,4}^{\rho,\sigma}$ transforms $B_3C_1C_5$ into

$$\xi_1' = -\xi_3, \qquad \xi_3' = \xi_1, \qquad \xi_2' = 2\rho\sigma\xi_2 + (\rho^2 - \sigma^2)\xi_4,$$

$$\xi_4' = (\rho^2 - \sigma^2)\xi_2 - 2\rho\sigma\xi_4, \qquad \xi_5' = -\xi_5,$$

which belongs to $K_8$ if and only if $\rho\sigma = 0$. According as $\sigma = 0$ or $\rho = 0$, it becomes $B_3C_1C_5$ or $B_3C_3C_0$, respectively. Hence if $O_{1,3}^{\lambda,\mu}O_{2,4}^{\rho,\sigma}$ belongs to $O_\Omega$ it is a $Q_{1,3}$, $Q_{1,3}B_3$, or the product of one of them by $C_2C_4$. Finally, $B_2$ does not transform $K_8$ into itself.

THEOREM. *Within $O_\Omega$, the group $K_8$ is self-conjugate only under*

$$H_{8(p^n-\nu)} = (Q_{1,3}^{\lambda,\mu}, B_3, G_{16}). \tag{64}$$

For $p^n = 3$ or 5, this group becomes $J_{32}^3$.

36. A substitution commutative with $H_8^3$ must be of the type $S$ of § 35. Now $O_{1,3}^{\lambda,\mu}O_{2,4}^{\rho,\sigma}$ evidently transforms $B_3C_1C_2 \equiv O_{1,3}^{\rho,-1}O_{2,4}^{\rho,-1}$ into itself. Hence it transforms into itself $B_3C_1C_4 \equiv B_3C_1C_2 \cdot C_2C_4$, $B_3C_2C_3 = B_3C_1C_2 \cdot C_1C_3$, $B_3C_3C_4 = B_3C_1C_2 \cdot C_1C_2C_3C_4$. Also, $B_2$ transforms $H_8^3$ into itself.

THEOREM. *Within $O_\Omega$, the group $H_8^3$ is self-conjugate only under*

$$H_{4(p^n-\nu)^2} = (Q_{1,3}^{\lambda,\mu}Q_{2,4}^{\rho,\sigma}, G_{64}). \tag{65}$$

For $p^n = 3$ or 5, this group becomes $G_{64}$.

37. A substitution $S$ commutative with $G_4^2$ must replace $\xi_5$ by $\pm\xi_5$ (Corollary II of § 22) and transform $C_1C_2$ into itself or $C_2C_4$. Hence $S = O_{1,2}O_{3,4}C$ or its product by $B_3$, respectively.

THEOREM. *Within $O_\Omega$, the group $G_4^2$ is self-conjugate only under*

$$H'_{4(p^n-\nu)^2} = (Q_{1,2}^{\lambda,\mu}Q_{3,4}^{\rho,\sigma}, G_{64}). \tag{65'}$$

For $p^n = 3$ or 5, this group becomes $G_{64}$.

38. The group $K_4'$ is certainly self-conjugate under $H_{96}$ of § 33. Within the latter $C_2C_4$, $C_2C_5$ and $C_4C_5$ are conjugate, and $H_{96}$ has substitutions which transform $C_2C_4$ into itself and $C_2C_5$ into $C_4C_5$. It thus remains to determine the substitutions $S$ commutative with each $C_2C_4$, $C_2C_5$, $C_4C_5$. Now $S = O_{1,3}^{\lambda,\mu}C$, where $C$ is a product of the $C_i$.

THEOREM. *Within $O_\Omega$, $K_4'$ is self-conjugate only under $H_{24(p^n-\nu)}$.*

39. A substitution commutative with $K_4'''$ and hence with $C_1C_3$ is either $S = O_{1,3}^{\lambda,\mu}O_{2,4,5}$ or $SC_1$. Now $S$ transforms $C_1C_2C_4C_5$ into

$$\xi_1' = (\mu^2 - \lambda^2)\xi_1 + 2\lambda\mu\xi_3, \qquad \xi_3' = 2\lambda\mu\xi_1 + (\lambda^2 - \mu^2)\xi_3,$$

$$\xi_2' = -\xi_2, \qquad \xi_4' = -\xi_4, \qquad \xi_5' = -\xi_5.$$

Hence $\lambda\mu = 0$ is the necessary and sufficient condition that the transform shall belong to $K_4'''$. The substitutions commutative with it are

$$CO_{2,4,5}, \qquad (\xi_1\xi_3)CO_{2,4,5} \qquad (C = I,\ C_1,\ C_3,\ C_1C_3).$$

The number of substitutions $O_{2,4,5}$ of determinant $\pm 1$ is $2(p^{2n}-1)p^n$, by *Linear Groups*, p. 160. Hence $\frac{1}{4}\cdot 8\cdot 2(p^{2n}-1)p^n$ substitutions of $O_\Omega$ are commutative with $K_4'''$.

THEOREM. *Within $O_\Omega$, $K_4'''$ is self-conjugate only under $G_{4p^n(p^{2n}-1)}$.*

COROLLARY. *Exactly $p^n(p^{2n}-1)(p^n-\nu)$ substitutions of $O_\Omega$ are commutative with $C_1C_3$.*

40. A substitution $S$ commutative with $J_8$ replaces $\xi_5$ by $\pm\xi_5$. If $S$ is of determinant $+1$ and is commutative with $B_3C_1C_2$ it has the form

$$K = \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & 0 \\ -\alpha_{13} & -\alpha_{14} & \alpha_{11} & \alpha_{12} & 0 \\ -\alpha_{23} & -\alpha_{24} & \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 0 & 0 & +1 \end{Vmatrix}.$$

If $K$ is commutative with $C_1C_2$, then $\alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = 0$. The resulting $2(p^n - \nu)$ substitutions are commutative with $B_3$ and hence with $J_8$ and all belong to $O_\Omega$. If $K$ transforms $C_1C_2$ into $B_3$ (and hence $B_3$ into $C_3C_4$ and hence $J_8$ into itself), then $\alpha_{13} = \alpha_{11}$, $\alpha_{14} = \alpha_{12}$, $\alpha_{23} = \alpha_{21}$, $\alpha_{24} = \alpha_{22}$. The orthogonal conditions then reduce to $\alpha_{21} = \pm\alpha_{12}$, $\alpha_{22} = \mp\alpha_{11}$, $\alpha_{11}^2 + \alpha_{12}^2 = \frac{1}{2}$. Denoting the resulting substitution by $K_\pm$, we have $K_- = K_+C_2C_4$. We proceed to show that $K_+$ (and hence $K_-$) does not belong to $O_\Omega$. Setting $\alpha_{11} = \alpha$ and $\alpha_{12} = \beta$, we have for $K_+$

$$[\alpha, \beta] = \begin{vmatrix} \alpha & \beta & \alpha & \beta & 0 \\ \beta & -\alpha & \beta & -\alpha & 0 \\ -\alpha & -\beta & \alpha & \beta & 0 \\ -\beta & \alpha & \beta & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \qquad (\alpha^2+\beta^2=\tfrac{1}{2}).$$

For $p^n = 3$, $[1, 1] = W^2(\xi_2\xi_3)C_2$,

$$[-1, -1] = [1, 1]\,C_1C_2C_3C_4,\ [1, -1] = W^2(\xi_2\xi_3\xi_4)(\xi_1\xi_2)C_1C_2C_4,$$

so that none of the $[\alpha, \beta]$ belong to $O_\Omega$.

For any $GF[p^n]$ in which $-1$ is the square of a mark $i$, we make the transformation of variables given in *Linear Groups*, p. 180, and get

| | $Y_{12}$ | $Y_{13}$ | $Y_{14}$ | $Y_{23}$ | $Y_{24}$ | $Y_{34}$ |
|---|---|---|---|---|---|---|
| $Y'_{12}$ | $\frac{1}{2}(1+\alpha)$ | $-\frac{1}{2}(\alpha+i\beta)$ | $\frac{1}{2}i\beta$ | $\frac{1}{2}i\beta$ | $-\frac{1}{2}(\alpha-i\beta)$ | $\frac{1}{2}(1-\alpha)$ |
| $Y'_{13}$ | $\frac{1}{2}(\alpha+i\beta)$ | $-i\beta$ | $\frac{1}{2}(\alpha-i\beta)$ | $\frac{1}{2}(\alpha-i\beta)$ | $\alpha$ | $-\frac{1}{2}(\alpha+i\beta)$ |
| $Y'_{14}$ | $\frac{1}{2}i\beta$ | $-\frac{1}{2}(\alpha-i\beta)$ | $\frac{1}{2}(1-\alpha)$ | $-\frac{1}{2}(1+\alpha)$ | $\frac{1}{2}(\alpha+i\beta)$ | $-\frac{1}{2}i\beta$ |
| $Y'_{23}$ | $\frac{1}{2}i\beta$ | $-\frac{1}{2}(\alpha-i\beta)$ | $-\frac{1}{2}(1+\alpha)$ | $\frac{1}{2}(1-\alpha)$ | $\frac{1}{2}(\alpha+i\beta)$ | $-\frac{1}{2}i\beta$ |
| $Y'_{24}$ | $\frac{1}{2}(\alpha-i\beta)$ | $\alpha$ | $-\frac{1}{2}(\alpha+i\beta)$ | $-\frac{1}{2}(\alpha+i\beta)$ | $i\beta$ | $-\frac{1}{2}(\alpha-i\beta)$ |
| $Y'_{34}$ | $\frac{1}{2}(1-\alpha)$ | $\frac{1}{2}(\alpha+i\beta)$ | $-\frac{1}{2}i\beta$ | $-\frac{1}{2}i\beta$ | $\frac{1}{2}(\alpha-i\beta)$ | $\frac{1}{2}(1+\alpha)$. |

The determinant (141) of *Linear Groups*, p. 154, here equals

$$\tfrac{1}{4}(1+2i\beta)(\alpha^2+i\beta+\beta^2)$$

and must be a square or zero. Applying $\alpha^2+\beta^2=\frac{1}{2}$, it reduces to

$$\tfrac{1}{2}\cdot\tfrac{1}{4}(1+2i\beta)^2.$$

By proper choice of $i$ as a root of $x^2=-1$, we can assume that $1+2i\beta \neq 0$. But 2 is a not-square. Hence none of the $[\alpha, \beta]$ belong to $O_\Omega$.

Finally, $C_1C_2$ of $J_8$ transforms $B_3C_1C_2$ into its inverse $B_3C_3C_4$.

THEOREM. *Within $O_\Omega$, the group $J_8$ is self-conjugate only under*

$$G_{8(p^n-\nu)} = (G_{32},\ Q^{\lambda,\mu}_{1,2}Q^{\lambda,\mu}_{3,4}). \tag{66}$$

COROLLARY. For $p^n = 3$ or $5$, $J_8$ is self-conjugate only under $G_{32}$.

41. A substitution $S$ commutative with $F'''_8$ replaces $\xi_5$ by $\pm\xi_5$. Then $S$ is commutative with $B_2C_1C_4$ if and only if it has the form

$$S_1 = \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ -\alpha_{12} & \alpha_{11} & \alpha_{14} & -\alpha_{13} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & 0 \\ \alpha_{32} & -\alpha_{31} & -\alpha_{34} & \alpha_{33} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{Vmatrix}.$$

Hence $S_1$ is commutative with the inverse $B_2C_2C_3$ of $B_2C_1C_4$. There are four further substitutions of period 4 in $F'''_8$: $B_3C_1C_2$, $B_3C_3C_4$, $B_4C_1C_3$, $B_4C_2C_4$. If $S_1$ is commutative with $B_3C_1C_2$, then $\alpha_{31} = -\alpha_{13}$, $\alpha_{32} = -\alpha_{14}$, $\alpha_{33} = \alpha_{11}$, $\alpha_{34} = \alpha_{12}$. The orthogonal conditions then reduce to $\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 + \alpha_{14}^2 = 0$. Hence by *Linear Groups*, p. 47, there are $p^{3n} - p^n$ substitutions $S'_1$ of determinant $+1$ commutative with $B_3C_1C_2$, and consequently commutative with

$$B_2C_1C_4 \cdot B_3C_1C_2 = B_4C_1C_3$$

and hence with the group $F'''_8$. If $S_1$ transforms $B_3C_1C_2$ into its inverse, $S_1 = S'_1C_1C_2$. If $S_1$ transforms $B_3C_1C_2$ into $B_4C_1C_3$, $S_1 = S'_1(\xi_2\xi_4\xi_3)$ and transforms $B_4C_1C_3$ into $B_3C_3C_4$. By symmetry there exist orthogonal substitutions of determinant $+1$ which transform $F'''_8$ into itself and transform $B_2C_1C_4$ into $B_3C_1C_2$ and are commutative with $B_4C_1C_3$. Hence there are $6 \cdot 4 \cdot (p^{3n} - p^n)$ orthogonal substitutions of determinant $+1$ which are commutative with $F'''_8$. Exactly half of these belong to $O_\Omega$, since $(\xi_1\xi_2)\,C_1$ transforms $B_2C_1C_4$, $B_3C_1C_2$ and $B_4C_1C_3$ into $B_2C_1C_4$, $B_4C_2C_4$ and $B_3C_1C_2$, respectively, and hence $F'''_8$ into itself.

THEOREM. *Within $O_\Omega$, $F'''_8$ is self-conjugate only under $G_{12\,p^n(p^{2n}-1)}$.*

42. The group $K_4^*$ contains $I$, $C_1C_3$, $B_3$ and $B_3C_1C_3$. Now $C_1C_5$ transforms $B_3$ into $B_3C_1C_3$, and $C_1C_3$ into itself. By § 32, $O_\Omega$ contains a substitution which transforms $C_1C_3$ and $B_3$ into each other. Hence the number of substitutions of $O_\Omega$ commutative with $K_4^*$ is 6 times the number commutative with each of its operators. If $(\alpha_{ij})$ is commutative with $C_1C_3$, then $\alpha_{12}$, $\alpha_{14}$, $\alpha_{15}$, $\alpha_{32}$, $\alpha_{34}$, $\alpha_{35}$, $\alpha_{21}$, $\alpha_{23}$, $\alpha_{41}$, $\alpha_{43}$, $\alpha_{51}$, $\alpha_{53}$ are all zero. If it is also commutative with $B_3$, then $\alpha_{31} = \alpha_{13}$, $\alpha_{33} = \alpha_{11}$, $\alpha_{42} = \alpha_{24}$, $\alpha_{44} = \alpha_{22}$, $\alpha_{45} = \alpha_{25}$, $\alpha_{54} = \alpha_{52}$. The resulting orthogonal substitutions are

$$(67) \qquad \begin{Vmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 & 0 \\ 0 & \alpha_{22} & 0 & \alpha_{24} & \alpha_{25} \\ \alpha_{13} & 0 & \alpha_{11} & 0 & 0 \\ 0 & \alpha_{24} & 0 & \alpha_{22} & \alpha_{25} \\ 0 & \alpha_{52} & 0 & \alpha_{52} & \alpha_{55} \end{Vmatrix} \quad \left\{ \begin{array}{l} \alpha_{11}^2 + \alpha_{13}^2 = 1,\ \alpha_{11}\alpha_{13} = 0 \\ \alpha_{22}^2 + \alpha_{24}^2 + \alpha_{25}^2 = 1,\ 2\alpha_{22}\alpha_{24} + \alpha_{25}^2 = 0 \\ 2\alpha_{52}^2 + \alpha_{55}^2 = 2\alpha_{25}^2 + \alpha_{55}^2 = 1 \\ \alpha_{52}(\alpha_{22} + \alpha_{24}) + \alpha_{25}\alpha_{55} = 0 \end{array} \right\}.$$

The condition that the determinant shall equal $+1$ is

$$(68) \qquad (\alpha_{22}-\alpha_{24})[\alpha_{55}(\alpha_{22}+\alpha_{24})-2\alpha_{25}\alpha_{52}]=1.$$

The conditions on $\alpha_{22}, \alpha_{24}, \alpha_{25}, \alpha_{52}, \alpha_{55}$ are seen to reduce to the following:

$$(69) \quad \alpha_{24}=\alpha_{22}\pm 1,\ \alpha_{52}=\pm\alpha_{25},\ \alpha_{55}=\mp 2\alpha_{22}-1,\ 2\alpha_{25}^2+(2\alpha_{22}\pm 1)^2=1.$$

By *Linear Groups*, p. 48, the last condition has $p^n+\nu$ sets of solutions $\alpha_{25}, 2\alpha_{22}\pm 1$, if 2 is a not-square and $\nu=\pm 1$ according as $p^n=4l\pm 1$. There are 4 sets of solutions of $\alpha_{11}^2+\alpha_{13}^2=1,\ \alpha_{11}\alpha_{13}=0$. Of the resulting $2\cdot 4\ (p^n+\nu)$ substitutions, half belong to $O_\Omega$, since but one of the pair $S$ and $S(\xi_1\xi_3)\,C_3$ does.

THEOREM. *Within $O_\Omega$, $K_4$ is self-conjugate only under $G_{24(p^n+\nu)}$.*

43. The group $T_8$ contains $C_1C_0$ and $C_3C_0$, but no further $C_iC_0$. Hence, as in the proof of Corollary III of § 22, a substitution $S$ commutative with $T_8$ must replace the pair $\xi_1, \xi_3$ by $\pm\xi_1, \pm\xi_3$ in some order. Hence $S$ is commutative with $C_1C_3$. If $S$ be commutative with $B_3$, it is of the form (67), of which $4(p^n+\nu)$ belong to $O_\Omega$. Then $S$ is commutative with $B_3C_1C_3$ and transforms $B_3C_1C_0$ into $B_3C_1C_0$ or $B_3C_3C_0$, since it transforms $C_1C_0$ and $C_3C_0$ amongst themselves. Next, $C_1C_5$ transforms $T_8$ into itself and $B_3$ into $B_3C_1C_3$, $B_3C_1C_0$ into $B_3C_3C_0$. Finally, $B_3$ and $B_3C_1C_0$ have different characteristic determinants.

THEOREM. *Within $O_\Omega$, $T_8$ is self-conjugate only under $G_{8(p^n+\nu)}$.*

44. Every orthogonal substitution commutative with $B_3C_1C_5$ has the form

$$(70) \qquad \begin{bmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 & 0 \\ 0 & \alpha_{22} & 0 & \alpha_{24} & \alpha_{25} \\ -\alpha_{13} & 0 & \alpha_{11} & 0 & 0 \\ 0 & \alpha_{24} & 0 & \alpha_{22} & -\alpha_{25} \\ 0 & \alpha_{52} & 0 & -\alpha_{52} & \alpha_{55} \end{bmatrix} \quad \left[\begin{array}{l} \alpha_{11}^2+\alpha_{13}^2=1,\ \alpha_{22}^2+\alpha_{24}^2+\alpha_{25}^2=1 \\ 2\alpha_{22}\alpha_{24}-\alpha_{25}^2=0 \\ 2\alpha_{52}^2+\alpha_{55}^2=2\alpha_{25}^2+\alpha_{55}^2=1 \\ \alpha_{22}\alpha_{52}-\alpha_{24}\alpha_{52}+\alpha_{25}\alpha_{55}=0 \end{array}\right].$$

The conditions on $\alpha_{22}, \alpha_{24}, \alpha_{25}, \alpha_{52}, \alpha_{55}$ and that for determinant $+1$ are seen to reduce to (69) if the sign of $\alpha_{24}$ is changed in the latter. Hence these conditions have $2(p^n+\nu)$ sets of solutions. Again, $\alpha_{11}^2+\alpha_{13}^2=1$ has $p^n-\nu$ sets of solutions. Hence exactly* $p^{2n}-1$ of the $2(p^{2n}-1)$ substitutions (70) of determinant $+1$ belong to $O_\Omega$.

Observing that $C_1C_5$ transforms $B_3C_1C_5$ into its inverse, we may state the

THEOREM. *Within $O_\Omega$, the group $(B_3C_1C_5)$ is self-conjugate only under a group $G_{2(p^{2n}-1)}$.*

* To make an explicit determination of them, we proceed as in *Linear Groups*, § 189. When $-1$ is the square of a mark $i$, (70) becomes

45. Since $L_8$ contains a single cyclic subgroup ($B_3C_1C_5$) of order 4, a substitution which transforms $L_8$ into itself must be of the form (70) or its product by $C_1C_5$. Now (70) transforms the substitution $C_1C_5$ of $L_8$ into

$$(71)\qquad \begin{bmatrix} \alpha_{13}^2-\alpha_{11}^2 & 0 & 2\alpha_{11}\alpha_{13} & 0 & 0 \\ 0 & 1-2\alpha_{25}^2 & 0 & 2\alpha_{25}^2 & k \\ 2\alpha_{11}\alpha_{13} & 0 & \alpha_{11}^2-\alpha_{13}^2 & 0 & 0 \\ 0 & 2\alpha_{25}^2 & 0 & 1-2\alpha_{25}^2 & -k \\ 0 & k & 0 & -k & 1-2\alpha_{55}^2 \end{bmatrix} \quad [k=\alpha_{52}(\alpha_{22}-\alpha_{24})-\alpha_{25}\alpha_{56}].$$

If (71) reduces to $C_1C_5$, then $\alpha_{13}=0$, $\alpha_{25}=0$, $\alpha_{55}=1$, so that (70) becomes $I$, $C_1C_3$, $C_2C_4$ or $C_1C_2C_3C_4$, in case it belongs to $O_\Omega$. If (71) reduces to $C_3C_5$, then $\alpha_{11}=0$, $\alpha_{25}=0$, $\alpha_{55}=-1$, so that (70) becomes $B_3C_iC_5$ or $B_3C_iC_0$ ($i=1,3$), in case it belongs to $O_\Omega$. The remaining substitutions of period 2 of $L_8$, other than $C_1C_3=(B_3C_1C_5)^2$, are $B_3$ and $B_3C_1C_3$. But (71) cannot reduce to either of these when 2 is a not-square. Now

$$(72)\qquad I,\ C_1C_3,\ C_2C_4,\ C_1C_2C_3C_4,\ B_3C_iC_5,\ B_3C_iC_0 \qquad (i=1,3),$$

together with their products by $C_1C_5$, give the 16 substitutions of $G_{16}'$.

$$\begin{bmatrix} \frac{1}{2}(1+\alpha_{11}) & -\frac{1}{2}\alpha_{13} & 0 & 0 & -\frac{1}{2}\alpha_{13} & \frac{1}{2}(1-\alpha_{11}) \\ \frac{1}{2}\alpha_{13} & \frac{1}{2}(\alpha_{11}+\alpha_{22}) & P_+ & P_- & \frac{1}{2}(\alpha_{11}-\alpha_{22}) & -\frac{1}{2}\alpha_{13} \\ 0 & P_\pm & A & B & -P_\pm & 0 \\ 0 & P_\mp & C & D & -P_\mp & 0 \\ \frac{1}{2}\alpha_{13} & \frac{1}{2}(\alpha_{11}-\alpha_{22}) & -P_+ & -P_- & \frac{1}{2}(\alpha_{11}+\alpha_{22}) & -\frac{1}{2}\alpha_{13} \\ \frac{1}{2}(1-\alpha_{11}) & \frac{1}{2}\alpha_{13} & 0 & 0 & \frac{1}{2}\alpha_{13} & \frac{1}{2}(1+\alpha_{11}) \end{bmatrix}$$

$P_\pm=\frac{1}{2}(\alpha_{22}\mp 1+i\alpha_{25})$,
$P_-=\frac{1}{2}(\alpha_{22}\mp 1-i\alpha_{25})$,
$A=\frac{1}{2}(\alpha_{22}\pm 2\alpha_{22}\pm i\alpha_{25}+i\alpha_{25}-1)$,
$B=\frac{1}{2}(\alpha_{22}\mp 2\alpha_{22}\pm i\alpha_{25}-i\alpha_{25}+1)$,
$C=\frac{1}{2}(\alpha_{22}\mp 2\alpha_{22}\mp i\alpha_{25}+i\alpha_{25}+1)$,
$D=\frac{1}{2}(\alpha_{22}\pm 2\alpha_{22}\mp i\alpha_{25}-i\alpha_{25}-1)$.

It is seen to be the second compound of

$$\Gamma=\begin{bmatrix} x & y & ry & -rx \\ z & w & rw & -rz \\ -rz & -rw & w & -z \\ rx & ry & -y & x \end{bmatrix} \qquad \left(r=\frac{-a_{12}}{1+a_{11}}=\frac{a_{11}-1}{a_{12}}\right),$$

if and only if the following conditions hold

$$xy=\frac{-P_\pm}{1+r^2},\quad xz=\frac{-P_+}{1+r^2},\quad xw=\frac{\alpha_{22}+1}{2(1+r^2)},\quad x^2=\frac{A}{1+r^2},\quad y^2=\frac{-B}{1+r^2},$$

$$zw=\frac{P_\mp}{1+r^2},\quad yw=\frac{P_-}{1+r^2},\quad yz=\frac{\alpha_{22}-1}{2(1+r^2)},\quad z^2=\frac{-C}{1+r^2},\quad w^2=\frac{D}{1+r^2}.$$

We have $1+r^2=2/(1+\alpha_{11})$. These conditions are seen to be compatible and to determine (except as to sign) marks $x$, $y$, $z$, $w$ of the field if and only if any non vanishing one of the last four fractions is a square. For example, $BC=\frac{1}{4}(1-\alpha_{22})^2$, $AD=\frac{1}{4}(1+\alpha_{22})^2$, $AB=-P_\pm^2$.

If $\alpha_{13}=0$, $\alpha_{11}=+1$, we take $r=0$. If $\alpha_{13}=0$, $\alpha_{11}=-1$, the formulæ fail, but the substitution (70) is then the product of the preceding by $C_1C_3$, so that one belongs to $O_\Omega$ if the other does.

THEOREM. *Within $O_\Omega$, the group $L_8$ is self-conjugate only under $G'_{16}$.*

46. The group $H^3_{16}$ contains 8 substitutions of period 4: $B_3C_iC_5$ and $B_3C_iC_0$ $(i=1,3)$, all of which are conjugate under $G_{64}$ (§ 6). A substitution which transforms $B_3C_1C_5$ into itself and $C_1C_2$ into a substitution of $H^3_{16}$ belongs to the set (72). Indeed, the conditions on (70) are

$$\alpha_{25}=0,\ \alpha_{11}=0,\ \alpha_{22}=0,\ \alpha_{55}=-1;\quad \text{or}\quad \alpha_{25}=0,\ \alpha_{13}=0,\ \alpha_{24}=0,\ \alpha_{55}=1.$$

THEOREM. *Within $O_\Omega$, the group $H^3_{16}$ is self-conjugate only under $G_{64}$.*

47. The only self-conjugate substitutions of period 4 of $F_{16}$ are $B_2C_1C_3$ and its inverse $B_2C_2C_4$ (§ 13). These must be transformed among themselves by any substitution commutative with $F_{16}$. Every substitution $S$ commutative with $B_2C_1C_3$ has the form

$$S=\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0\\ -\alpha_{12} & \alpha_{11} & -\alpha_{14} & \alpha_{13} & 0\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & 0\\ -\alpha_{32} & \alpha_{31} & -\alpha_{34} & \alpha_{33} & 0\\ 0 & 0 & 0 & 0 & \pm 1 \end{vmatrix}.$$

The further substitutions of period 4 of $F_{16}$ are $B_2C_1C_4$ and $B_2C_2C_3$, $B_3C_1C_2$ and $B_3C_3C_4$, $B_4C_1C_3$ and $B_4C_2C_4$, the two of each pair being conjugate within $G_{32}$, under which $F_{16}$ is self-conjugate (§ 10).

If $S$ is commutative with $B_2C_1C_4$, then $\alpha_{13}$, $\alpha_{14}$, $\alpha_{31}$, $\alpha_{32}$ are zero, so that $S=O^{\alpha_{11},\,\alpha_{12}}_{1,2}O^{\alpha_{33},\,\alpha_{34}}_{3,4}$ if it is orthogonal and of determinant $+1$. If further $S$ be commutative with $B_3C_1C_2$ and hence with $F_{16}$, then $\alpha_{33}=\alpha_{11}$, $\alpha_{34}=\alpha_{12}$. But if $S$ transforms $B_3C_1C_2$ into $B_4C_1C_3$, then $\alpha_{33}=\alpha_{12}$, $\alpha_{34}=-\alpha_{11}$, so that $S=O^{\alpha_{11},\,\alpha_{12}}_{1,2}O^{\alpha_{11},\,\alpha_{12}}_{3,4}(\xi_3\xi_4)C_3$ and hence is not in $O_\Omega$. Hence $O^{\alpha_{11},\,\alpha_{12}}_{1,2}O^{\alpha_{11},\,\alpha_{12}}_{3,4}$ and its product by $C_1C_2$ are the only substitutions $S$ of $O_\Omega$ which are commutative with $F_{16}$ and $B_2C_1C_4$. Their products by $B_3$ are the only ones transforming $B_2C_1C_4$ into $B_2C_2C_3$.

If an orthogonal substitution of the form $S$ transforms $B_2C_1C_4$ into $B_3C_1C_2$, it has the form

$$S'=\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0\\ -\alpha_{12} & \alpha_{11} & -\alpha_{14} & \alpha_{13} & 0\\ -\alpha_{12} & \alpha_{11} & \alpha_{14} & -\alpha_{13} & 0\\ -\alpha_{11} & -\alpha_{12} & \alpha_{13} & \alpha_{14} & 0\\ 0 & 0 & 0 & 0 & \pm 1 \end{vmatrix} \qquad \left(\begin{matrix}\alpha_{11}^2+\alpha_{12}^2=\frac{1}{2}\\ \alpha_{13}^2+\alpha_{14}^2=\frac{1}{2}\end{matrix}\right).$$

Its determinant equals $\pm 4(\alpha_{11}^2+\alpha_{12}^2)(\alpha_{13}^2+\alpha_{14}^2)$. We therefore take $\pm 1=+1$. Then $S'$ transforms $B_3C_1C_2$ into

$$\begin{vmatrix} 0 & \rho & 0 & \sigma & 0 \\ -\rho & 0 & -\sigma & 0 & 0 \\ 0 & \sigma & 0 & -\rho & 0 \\ -\sigma & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad \begin{pmatrix} \rho = 2a_{11}a_{14} - 2a_{12}a_{13} \\ \sigma = -2a_{11}a_{13} - 2a_{12}a_{14} \end{pmatrix}.$$

Then $\rho^2 + \sigma^2 = 1$. This belongs to $F_{16}$ (and consequently $S'$ transforms $F_{16}$ into itself) only when $\rho\sigma = 0$. If $\sigma = 0$, it becomes $B_2C_1C_4$ or $B_2C_2C_3$. If $\rho = 0$, it becomes $B_4C_1C_3$ or $B_4C_2C_4$. Since 2 is a not-square, the conditions on $S'$ show that $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}$ all differ from 0. Hence $\rho = 0$ gives $\alpha_{13} = \pm\alpha_{11}$, $\alpha_{14} = \pm\alpha_{12}$, while $\sigma = 0$ gives $\alpha_{13} = \pm\alpha_{12}$, $\alpha_{14} = \mp\alpha_{11}$. From the remark at the end of the section it follows * indirectly that exactly half of the resulting substitutions belong to $O_\Omega$.

If an orthogonal substitution of the form $S$ transforms $B_2C_1C_4$ into $B_4C_1C_3$ then $S = S'(\xi_3\xi_4)C_3$.

The total number of orthogonal substitutions $S$ of determinant $+1$ which transforms $F_{16}$ into itself is therefore $6\cdot4\cdot(p^n - \nu)$. These, together with their products by $C_1C_3$ (which transforms $F_{16}$ into itself and $B_2C_1C_3$ into its inverse $B_2C_2C_4$), give all of determinant $+1$ which transforms $F_{16}$ into itself. But $(\xi_1\xi_2)C_1$ transforms $F_{16}$ into itself. Hence exactly $6\cdot4\cdot(p^n - \nu)$ belong to $O_\Omega$.

THEOREM. *Within $O_\Omega$, $F_{16}$ is self-conjugate only under $G_{24(p^n-\nu)}$.*

Another proof follows from the results of § 26. The substitutions of $O_\Omega$ commutative with $B_2C_1C_3$ are found from those commutative with $B_3C_1C_4$ by transformation by $(\xi_2\xi_3\xi_4)$. From (55) we thus get

$$(73)\quad B_i,\ B_iC_1C_2,\ B_iC_3C_4,\ B_iC_1C_2C_3C_4,\ B_jC_1C_3,\ B_jC_1C_4,\ B_jC_2C_3,\ B_jC_2C_4 \qquad (i = 1, 3;\ j = 2, 4).$$

Hence, for $p^n = 3$, these and their products by $W(\xi_2\xi_4\xi_3)$ and by its inverse give all the substitutions commutative with $B_2C_1C_3$. Inversely, they transform $F_{16}$ into itself. For $p^n = 5$, the 12 types $S'$ with a single vanishing $\alpha_{ij}$ are seen to be excluded as in § 27. Consider next $\Sigma^*$, the transform of $S'$ by $(\xi_2\xi_3\xi_4)$, where $S'$ is the substitution of § 11 subject to the conditions (56). We find that $\Sigma^*$ transforms $C_1C_2$ and $B_3$ into respectively

* To give a direct proof for $p^n = 3$, we note a substitution given by the lower signs is the product of $C_3C_4$ and that given by the upper signs. For $\alpha_{13} = \alpha_{11} = +1$, $\alpha_{14} = \alpha_{12} = +1$, $S' = W^2(\xi_2\xi_3\xi_4)$; for $\alpha_{13} = \alpha_{11} = +1$, $\alpha_{14} = \alpha_{12} = -1$, $S' = W^2(\xi_2\xi_3\xi_4)C_2C_4B_2$; for $\alpha_{13} = \alpha_{12} = 1$, $\alpha_{14} = -\alpha_{11} = -1$, $S' = C_2C_3W(\xi_2\xi_4\xi_2)(\xi_3\xi_4)C_2C_3C_4 \equiv S''$; for $\alpha_{13} = \alpha_{12} = 1$, $\alpha_{14} = -\alpha_{11} = 1$, $S' = C_2C_3S''C_1C_4$. All other cases follow at once from these.

$$\begin{bmatrix} 0 & 0 & \rho & \sigma & 0 \\ 0 & 0 & -\sigma & \rho & 0 \\ \rho & -\sigma & 0 & 0 & 0 \\ \sigma & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 & \mu & 0 \\ 0 & \lambda & -\mu & 0 & 0 \\ 0 & -\mu & -\lambda & 0 & 0 \\ \mu & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{cases} \rho = 2\alpha_{12}\alpha_{22} + 2\alpha_{11}\alpha_{13}\alpha_{14}\alpha_{22} \\ \sigma = 2\alpha_{11}\alpha_{13} + 2\alpha_{12}\alpha_{14} \\ \lambda = 2\alpha_{11}\alpha_{12} + 2\alpha_{13}\alpha_{14} \\ \mu = 2\alpha_{11}\alpha_{14} - 2\alpha_{12}\alpha_{13} \end{cases}.$$

Since $\rho^2 + \sigma^2 \equiv 1$, either $\rho = 0$, whence the first substitution is either $B_4C_1C_4$ or $B_4C_2C_3$, or $\sigma = 0$, whence it is either $B_3$ or $B_3C_1C_2C_3C_4$. Since $\lambda^2+\mu^2\equiv 1$, either $\lambda = 0$ and the second substitution is either $B_4C_1C_4$ or $B_4C_2C_3$, or $\mu = 0$ and it is either $C_1C_2$ or $C_3C_4$. The resulting substitutions all belong to $F_{16}$. But $B_2C_1C_3$, $C_1C_2$ and $B_3$ generate $F_{16}$. Hence each of the 32 substitutions $\Sigma^*$ transforms $F_{16}$ into itself. These together with the 16 substitutions (73) give all the 48 substitutions of $O_\Omega$ which transform $F_{16}$ and $B_2C_1C_3$ each into itself. But $B_2$ transforms $F_{16}$ into itself and $B_2C_1C_3$ into its inverse. *Hence $F_{16}$ is self-conjugate only under the group $(G_{32}, \Sigma^*)$ of order* 96.

THE UNIVERSITY OF CHICAGO,
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# ON CERTAIN INVARIANTS OF TWO TRIANGLES*

BY

JOHN GALE HUN

The purposes of this paper are to express some of the simpler invariant relations of two triangles in terms of three fundamental invariants, and to consider the case of two triangles apolar both ways. The fundamental invariants are derived by a method first presented by HILBERT in a letter to HERMITE. †

## § 1. *The three fundamental invariants.*

With two plane curves, one of $n$th order $\alpha_x^n = 0$ and the other of $n$th class $a_\xi^n = 0$, there is associated the connex

$$\alpha_y a_\eta \alpha_a^{n-1} = 0,$$

found by acting on the line curve with the polar of the point $y$ as to the point curve. ‡ The fixed points of this connex are given by the equations

$$a_i \alpha_y \alpha_a^{n-1} = \lambda y_i, \tag{1}$$

which are consistent if

$$\Delta(\lambda) \equiv \begin{vmatrix} a_1\alpha_1 a_a^{n-1} - \lambda & a_2\alpha_1 a_a^{n-1} & a_3\alpha_1 a_a^{n-1} \\ a_1\alpha_2 a_a^{n-1} & a_2\alpha_2 a_a^{n-1} - \lambda & a_3\alpha_2 a_a^{n-1} \\ a_1\alpha_3 a_a^{n-1} & a_2\alpha_3 a_a^{n-1} & a_3\alpha_3 a_a^{n-1} - \lambda \end{vmatrix} = 0. \tag{2}$$

The coefficients of this cubic are invariants of the two curves.

We shall consider the case where $n = 3$, and the point and line cubics degenerate respectively into a 3-line and a 3-point. Let the 3-point be

$$a_\xi b_\xi c_\xi = 0,$$

and the 3-line

$$\alpha_x \beta_x \gamma_x = 0.$$

We shall make use of the following notation:

$$b_\beta c_\gamma + b_\gamma c_\beta = (bc/\beta\gamma),$$

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† Journal de Mathématiques, ser. 4, vol. 4 (1886), p. 249.

‡ HILBERT, loc. cit.

$$\sum_a (bc/\beta\gamma)\, a_i = (\beta\gamma)_i,$$

$$\sum_a (\beta\gamma)_i \alpha_j = \sum_a (bc)_j a_i = B_{ji}.$$

The equation (2) then becomes

$$\Delta(\lambda) = \begin{vmatrix} B_{11}-\lambda & B_{12} & B_{13} \\ B_{21} & B_{22}-\lambda & B_{23} \\ B_{31} & B_{32} & B_{33}-\lambda \end{vmatrix} = 0. \tag{3}$$

This equation may be written in the form

$$\lambda^3 + I_1\lambda^2 + I_2\lambda + I_3 = 0, \tag{4}$$

where

$$I_1 = -\sum B_{11},$$

$$I_2 = \sum (B_{22}B_{33} - B_{23}B_{32}),$$

$$I_3 = -\begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix}$$

$$= -|\alpha\beta\gamma| \cdot |abc| \cdot \begin{vmatrix} (bc/\beta\gamma) & (bc/\gamma\alpha) & (bc/\alpha\beta) \\ (ca/\beta\gamma) & (ca/\gamma\alpha) & (ca/\alpha\beta) \\ (ab/\beta\gamma) & (ab/\gamma\alpha) & (ab/\alpha\beta) \end{vmatrix} = \Delta DN.$$

## § 2. *Invariants of the connex.**

The connex set up by the two triangles is

$$\sum_{i,j} B_{ij} y_i \eta_j = 0.$$

This gives rise to the two collineations

$$Y: \qquad y_i' = \sum_j B_{ji} y_j,$$

and

$$H: \qquad \eta_i' = \sum B_{ij} \eta_j$$

By taking the triangle of reference as the fixed triangle of $Y$, it is at once seen that $Y$ is the inverse of $H$.

*Cf. CLEBSCH, *Géométrie*, French translation, vol. 1, p. 311ff.

Let the reference triangle be now taken as that of the 3-point. Then

$$B_{ij} = \sum_{\alpha} (\beta_\kappa \gamma_l + \beta_l \gamma_\kappa) \alpha_i \qquad (j \neq \kappa \neq l),$$

and $Y$ sends the point $(1, 0, 0)$ into $(B_{11}, B_{12}, B_{13})$. This point lies on the line $(1, 0, 0)$ if $B_{11}$ is zero. But

$$-\tfrac{1}{3} I_1 = B_{11} = B_{22} = B_{33} = \sum_{\alpha} (\beta_2 \gamma_3 + \beta_3 \gamma_1) \alpha_1.$$

If then $I_1 = 0$, $Y$ sends the 3-point into an inscribed 3-point. Evidently, this is also the condition that $H$ sends the 3-line into a circumscribed 3-line. Since $H$ is the inverse of $Y$, it follows that $Y$ and $H$ respectively send the 3-point and 3-line into inscribed and circumscribed triangles.* A collineation whose first invariant vanishes may be said to be *normal*.

If $I_2 = 0$, the sum of the minors of the elements of the principal diagonal in the determinant of the connex is zero. The collineations $Y$ and $H$ may then be said to be sub-normal. They have respectively the property of sending certain triangles into circumscribed and inscribed triangles. It is, however, easy to see that neither the 3-point nor the 3-line is such a triangle.

If $I_3 = 0$, the determinant of the connex is zero. In this case zero is a root of (4), and this value of $\lambda$, instead of giving a proper fixed point of $Y$, will give a point $y_0$ which $Y$ explodes; i. e., which $Y$ sends anywhere. Similarly, there is a line $\eta_0$ which $H$ explodes. In general, $y_0$ and $\eta_0$ are not incident point and line, and hence may be taken as opposite point and line of the reference triangle. Then

$$B_{ii} = B_{1i} = 0 \qquad (i = 1, 2, 3).$$

The collineation $Y$ sends all points on a line through $y_0$ into the same point on $\eta_0$. Among these lines there are two all points of which will be sent into their intersections with $\eta_0$. These intersections are then the proper fixed points of $Y$; and the lines joining them to $y_0$ are the proper fixed lines of $H$. Upon taking the two fixed points of $Y$ as the remaining reference points, the collineations become

$$Y: \qquad y_1' = 0, \qquad y_2' = B_{22} y_2, \qquad y_3' = B_{33} y_3,$$

$$H: \qquad \eta_1' = 0, \qquad \eta_2' = B_{22} \eta_2, \qquad \eta_3' = B_{33} \eta_3.$$

Evidently then, if $Y$ sends $y$ into $y'$, $H$ sends any line through $y'$ into the join of $y$ and $y_0$. Since $I_3 = \Delta D N$, the collineations may be brought into these forms when the 3-point or 3-line degenerates. It may be shown that, if $\Delta = 0$, $y_0$ is the point in which $\alpha$, $\beta$ and $\gamma$ meet; and, if $D = 0$, that $y_0$ is the line in which $a$, $b$ and $c$ lie.

* Cf. PASCH, Mathematische Annalen, vol. 23 (1884), p. 426.

The invariants of the connex yield then the following invariants of the triangles:

$$I_1(3,3), \qquad I_2(6,6), \qquad N(6,6), \qquad \Delta(0,3), \qquad D(3,0),$$

where the numbers in the parentheses indicate the collective degrees in the Roman and Greek letters respectively.

From these five invariants can be built three absolute invariants,

$$\frac{N}{I_2}, \qquad \frac{I_1}{\Delta D}, \qquad \frac{I_1^2}{I_2}.$$

Now two triangles have $12 - 8 = 4$ absolute invariants. Thus it appears that (as we should expect) all invariants of the triangles are not obtainable by a consideration of the connex alone.

### § 3. *Special forms taken by the fundamental invariants.*

In order to determine the invariants of the 3-point and 3-line it will be found to be convenient to know the forms that $I_1$, $I_2$, $\Delta D$ and $N$ assume when the 3-line is taken as the reference triangle and the points $b$ and $c$ of the 3-point as the absolute.

We have for this special case, by direct substitution and reduction,

$$\begin{aligned} I_1 &= 12 \sum c_1 a_1, \\ I_2 &= 16 \sum (2c_1^2 - 1) a_1^2 + 2(2 - c_2 c_3) a_2 a_3, \\ \Delta D &= 4i \sum a_1, \\ N &= 16i \sum c_1 a_1^2 - (c_2 + c_3) a_2 a_3, \\ \Delta D I_1 &= 48i \sum c_1 a_1^2 + (c_2 + c_3) a_2 a_3, \end{aligned} \tag{5}$$

where the $c$'s are the cotangents of the interior angles of the reference triangle.

### § 4. *The interchange of the* 3*-point and* 3*-line.*

If the vanishing of some invariant $I$ be found to express the condition that the 3-point and 3-line have a certain projective property, then the vanishing of the invariant formed by substituting in $I$, for each Roman letter the minor in $\Delta$ of the corresponding Greek letter, and for each Greek letter the minor in $D$ of the corresponding Roman letter, must express the condition that the meets of the 3-line and the joins of the 3-point have the same projective property. The algebra involved in making this substitution being extremely long, we shall prove one formula only, and in the others merely give the results obtained.

By definition

$$I_2 = \sum_1^3 B_{22} B_{33} - B_{23} B_{32},$$

and

$$B_{ij} = \sum_a \sum_\alpha (bc/\beta\gamma) a_j \alpha_i.$$

Then

$$\begin{aligned} B_{22} B_{33} &= \Big[\sum_a \sum_\alpha (bc/\beta\gamma) a_2 \alpha_2\Big] \cdot \Big[\sum_a \sum_\alpha (bc/\beta\gamma) a_3 \alpha_3\Big] \\ &= \sum_a \sum_\alpha [(bc/\beta\gamma)^2 a_2 \alpha_2 a_3 \alpha_3 + (bc/\gamma\alpha)(bc/\alpha\beta)\{a_2\beta_2 a_3\gamma_3 + a_2\gamma_2 a_3\beta_3\} \\ &\qquad + (ca/\beta\gamma)(ab/\beta\gamma)\{b_2\alpha_2 c_3\alpha_3 + c_2\alpha_2 b_3\alpha_3\} \\ &\qquad + (ca/\gamma\alpha)(ab/\alpha\beta)\{b_2\beta_2 c_3\gamma_3 + c_2\gamma_2 b_3\beta_3\} \\ &\qquad + (ca/\alpha\beta)(ab/\gamma\alpha)\{b_2\gamma_2 c_3\beta_3 + b_3\gamma_3 c_2\beta_2\}]. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_1^3 B_{22} B_{33} = \sum_a \sum_\alpha [&(bc/\gamma\alpha)(bc/\alpha\beta)\, a_\beta a_\gamma + (ca/\beta\gamma)(ab/\beta\gamma) b_\alpha c_\alpha \\ &+ (ca/\gamma\alpha)(ab/\alpha\beta)\, b_\beta c_\gamma + (ca/\alpha\beta)(ab/\gamma\alpha) b_\gamma c_\beta \\ &+ \sum_1^3 \{(bc/\beta\gamma)^2 a_2\alpha_2 a_3\alpha_3 - (bc/\gamma\alpha)(bc/\alpha\beta) a_1^2 \beta_1\gamma_1 \\ &- (ca/\beta\gamma)(ab/\beta\gamma) b_1 c_1 \alpha_1^2 - (ca/\gamma\alpha)(ab/\alpha\beta) b_1 c_1 \beta_1\gamma_1 \\ &- (ca/\alpha\beta)(ab/\gamma\alpha) b_1 c_1 \beta_1\gamma_1\}]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sum_1^3 B_{23} B_{32} = \sum_a \sum_\alpha [&(bc/\gamma\alpha)(bc/\alpha\beta)\, a_\beta a_\gamma + (ca/\beta\gamma)(ab/\beta\gamma) b_\alpha c_\alpha \\ &+ (ca/\gamma\alpha)(ab/\alpha\beta)\, b_\gamma c_\beta + (ca/\alpha\beta)(ab/\gamma\alpha) b_\beta c_\gamma \\ &+ \sum_1^3 \{(bc/\beta\gamma)^2 a_2\alpha_2 a_3\alpha_3 - (bc/\gamma\alpha)(bc/\alpha\beta) a_1^2 \beta_1\gamma_1 \\ &- (ca/\beta\gamma)(ab/\beta\gamma) b_1 c_1 \alpha_1^2 - (ca/\gamma\alpha)(ab/\alpha\beta) b_1 c_1 \beta_1\gamma_1 \\ &- (ca/\alpha\beta)(ab/\gamma\alpha) b_1 c_1 \beta_1\gamma_1\}]. \end{aligned}$$

We have then

$$I_2 = \sum_a \sum_\alpha a_\alpha (b_\beta c_\gamma - b_\gamma c_\beta)[a_\alpha (b_\beta c_\gamma - b_\gamma c_\beta) - a_\beta (b_\gamma c_\alpha - b_\alpha c_\gamma) - a_\gamma (b_\alpha c_\beta - b_\beta c_\alpha)]$$

Let now,

$$\alpha_1' = b_2 c_3 - b_3 c_2, \text{ etc.}$$

Then

$$a_{\alpha'}' = b_\beta c_\gamma - b_\gamma c_\beta.$$

Hence,

$$(b_\beta c_\gamma - b_\gamma c_\beta)' = a_\alpha [a_\alpha(b_\beta c_\gamma - b_\gamma c_\beta) + a_\beta(b_\gamma c_\alpha - b_\alpha c_\gamma) + a_\gamma(b_\alpha c_\beta - b_\beta c_\alpha)]$$
$$= \Delta D a_\alpha .$$

Substituting these values in the expression for $I_2'$, we have

$$I_2' = \Delta^2 D^2 \sum_\alpha \sum_\alpha a_\alpha (b_\beta c_\gamma - b_\gamma c_\beta)[a_\alpha(b_\beta c_\gamma - b_\gamma c_\beta) - a_\beta(b_\gamma c_\alpha - b_\alpha c_\gamma) - a_\gamma(b_\alpha c_\beta - b_\beta c_\alpha)]$$
$$= \Delta^2 D^2 I_2 .$$

By similar calculation we find the following values for the invariants $I_1'$, $\Delta'$, $D'$ and $N'$:

$$(6) \qquad \begin{cases} I_1' = -\tfrac{1}{2}(\Delta D I_1 - 9N), \\ \Delta' D' = \Delta^2 D^2, \\ I_2' = \Delta^2 D^2 I_2, \\ N' = \dfrac{\Delta^2 D^2}{6}(\Delta D I_1 + 3N). \end{cases}$$

## § 5. *Invariant relations of the* 3*-point and* 3*-line.*

We are now in a position to express the conditions for certain invariant relations of the 3-point and 3-line.

I. *The 3-point degenerates if* $D = 0$.

II. *The 3-line degenerates if* $\Delta = 0$.

III. *Let the 3-point be apolar to the 3-line.*

Then $a_\xi b_\xi c_\xi$ acting on $\alpha_x \beta_x \gamma_x$ must be zero. That is,

$$I_1 = 3 \sum_\alpha (bc/\beta\gamma) a_\alpha = 0 .$$

The required condition is then $I_1 = 0$.

IV. *Let the joins of the* 3*-point be apolar to the meets of the* 3*-line.*

Then the lines $ca$ and $ab$ are apolar to the polar conic of $bc$ as to the meets of the 3-line. That is, if $b$ and $c$ are the absolute $I$, $J$ and the 3-line the triangle of reference, the lines $ab$ and $ac$ are apolar to the maximum inscribed ellipse, $\sum \xi_2 \xi_3 = 0$. In other words, the tangents from $a$ to this conic are perpendicular, and hence $a$ lies on the director circle.

If we define the *intermediate* of two line conics as the locus of points, $a$, such that the lines of the two conics through them form harmonic pairs, the director circle is the intermediate of the ellipse and the absolute. The intermediate * of two conics, $u_\xi^2 = 0$ and $v_\xi^2 = 0$, is

* SALMON, *Conic Sections*, tenth ed., pp. 306–7.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}^2 = \sum (u_{22}v_{33} + u_{33}v_{22} - u_{23}v_{23})a_1^2 + (u_{31}v_{12} + u_{12}v_{31} - u_{11}v_{23} - u_{23}v_{11})a_2a_3 = 0.$$

If then

$$u_\xi^2 \equiv \sum \xi_2\xi_3 \qquad \text{and} \qquad v_\xi^2 \equiv \sum (c_2 + c_3)\xi_1^2 - 2c_1\xi_2\xi_3,$$

the director circle becomes

$$\sum c_1a_1^2 - 2(c_2 + c_3)a_2a_3 = 0.$$

From the equations (5) we have

$$\Delta DI_1 - 9N \equiv -96i\sum c_1a_1^2 - 2(c_2 + c_3)a_2a_3.$$

The required condition is then

$$\Delta DI_1 - 9N = 0.$$

V. *Let there be a point conic circumscribed to 3-point and apolar to the 3-line.*

In the special case of § 3, the conic becomes the apolar circle, whose equation is

$$\sum c_1a_1^2 = 0.$$

But, from (5), we have

$$\Delta DI_1 + 3N = 96i\sum c_1a_1^2.$$

The required condition is then

$$\Delta DI_1 + 3N = 0.$$

VI. *Let there be a line conic touching the joins of the 3-point and apolar to the 3-line.*

The equations (6) applied to the invariant $V$ give the required invariant, to within powers of $\Delta$ and $D$, in the form $N = 0$. To determine the proper degree we proceed as follows. Let accented Greek letters denote the minors of the corresponding letters in $\Delta$.

Any line conic apolar to the 3-line is given by

$$\sum m_1(\alpha_\xi')^2 = 0,$$

and any line conic inscribed to the 3-point by

$$\sum m_4b_\xi c_\xi = 0.$$

We wish then these two conics to be identical. Upon equating the coefficients, we have six homogeneous equations in the six $m$'s. They are consistent

if their determinant vanishes. This determinant is then the invariant we wish to find. It is evidently of the sixth degree in the Roman letters and of the twelfth in the Greek letters. The required invariant is then

$$\Delta^2 N = 0.$$

This invariant may also be easily found directly. In fact, in the special case of § 3, we have only to make $a$ the focus of an apolar parabola. The joins of the middle points of a triangle are lines of all parabolas apolar to the triangle. But a circle, circumscribing a triangle formed by three tangents to a parabola, passes through the focus.*

Then $a$ must lie on the circle passing through the middle points of the sides of the 3-line; i. e., on the Feuerbach circle, whose equation is

$$N = 16i \sum c_1 a_1^2 - (c_2 + c_3) a_2 a_3 = 0.$$

The remaining factor in the invariant is found as before.

Since the invariant contains the factor $\Delta$, it can only put two conditions on a conic to make it apolar to a degenerate 3-line. This may also be easily shown directly.

VII. *Let there be a conic circumscribed to the* 3*-point and to the* 3*-line.*

In the case of § 3, the conic becomes the circumcircle, whose equation is

$$\sum (c_2 + c_3) a_2 a_3 = 0.$$

From (5) we have

$$\Delta D I_1 - 3N = 96i \sum (c_2 + c_3) a_2 a_3.$$

This invariant is then a factor in the required condition, which an investigation of the degree shows to be

$$\Delta^2 (\Delta D I_1 - 3N) = 0.$$

VIII. *Let there be a conic inscribed to the* 3*-line and to the* 3*-point.*

The required condition is found by an interchange of the Greek and Roman letters in the preceding condition. It is then

$$D^2 (\Delta D I_1 - 3N) = 0.$$

If neither the 3-point nor the 3-line degenerates, the conditions VII and VIII are the same. We have then an immediate proof of the well-known theorem † that the vertices of two triangles, circumscribed to a conic, lie on a conic.

IX. *Let there be a line conic apolar to the* 3*-line and to the* 3*-point.*

The line conic apolar to the 3-point is of the form

*SALMON, *Conic Sections*, tenth ed., p. 207.

†SALMON, *Conic Sections*, tenth ed., pp. 320 and 343.

$$\sum m_1 (a_\xi)^2 = 0,$$

and a line conic apolar to the 3-line is of the form

$$\sum m_1 (\alpha'_\xi)^2 = 0.$$

These conics are identical if

$$\sum m_1 a_i a_j = \sum m_1 \alpha'_i \alpha'_j .$$

The condition for consistency of these six equations is the vanishing of the same determinant as that found in investigating the degree of the invariant VII.

The required condition is then

$$\Delta^2 (\Delta D I_1 - 3N) = 0.$$

X. *Let there be a point conic apolar to the* 3*-line and to the* 3*-point.*

The required invariant condition is found from the preceding by an interchange of the Greek and Roman letters. It is then

$$D^2 (\Delta D I_1 - 3N) = 0.$$

It is obvious that the substitution (6) applied to IX and X must merely reproduce these conditions. This is at once seen to be the case.

XI. *Let there be a point* $y$ *such that its polar conic as to the* 3*-line is apolar to the* 3*-point.*

Referring to § 2, we see that $y$ is exploded by the collineation $Y$. The condition that there may be such a point is then

$$I_3 = 0.$$

XII. *Let there be a point* $y$ *such that its polar conic as to the joints of the* 3*-point is apolar to the meets of the* 3*-line.*

By applying the substitution (6) to the preceding invariant, and investigating the degree, we find the required condition in the form

$$\Delta^2 D^2 (\Delta D I_1 + 3N) = 0.$$

XIII. *Let the three polar lines, as to the* 3*-line, of the points of the* 3*-point, taken two at a time, meet in a point.*

The polar line of $b_\xi c_\xi$ as to the 3-line is

$$\sum_a (bc/\beta\gamma)\alpha_x \equiv \sum_{1,2,3} (bc)_1 x_1 = 0.$$

These three lines meet in a point if

$$|(bc)_1, (ca)_2, (ab)_3| \equiv \Delta N = 0.$$

This then gives a simple projective definition of the Feuerbach conic of two points and a triangle, where we mean by the Feuerbach conic that conic which

will project into the Feuerbach circle when the two points are projected into the absolute.

XIV. *Let there be a collineation, having the* 3*-line as its fixed triangle, which sends the* 3*-point into an inscribed* 3*-point.*

If the 3-line be the triangle of reference, the collineation is of the form

$$x_i' = k_i x_i.$$

This collineation sends $a$ into a point on the join of $b$ and $c$ if

$$\sum a_1 A_1 k_1 = \begin{vmatrix} a_1 k_1 & a_2 k_2 & a_3 k_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Similarly, the conditions that $b$ and $c$ are sent into points on the joins of $c$ and $a$, and $a$ and $b$ respectively are

$$\sum b_1 B_1 k_1 = 0 \qquad \text{and} \qquad \sum c_1 C_1 k_1 = 0.$$

Adding these three equations, we have, as the condition upon the collineation itself,

$$k_1 + k_2 + k_3 = 0.$$

Elimination of the $k$'s gives, as the condition the 3-point must satisfy,

$$\begin{vmatrix} a_1 A_1 & a_2 A_2 & a_3 A_3 \\ b_1 B_1 & b_2 B_2 & b_3 B_3 \\ c_1 C_1 & c_2 C_2 & c_3 C_3 \end{vmatrix}$$

$$= \sum a_1 (b_2 c_3 - b_3 c_2) [b_2 c_3 (c_3 a_1 - c_1 a_3)(a_1 b_2 - a_2 b_1) - b_3 c_2 (c_1 a_2 - c_2 a_1)(a_3 b_1 - a_1 b_3)]$$

$$= \sum a_1 (b_2 c_3 - b_3 c_2) [a_1^2 (b_2^2 c_3^2 - b_3^2 c_2^2) + a_2 a_3 b_1 c_1 (b_2 c_3 - b_3 c_2) - a_3 a_1 (b_2^2 c_3 c_1 - b_3 b_1 c_2^2) - a_1 a_2 (b_1 b_2 c_3^2 - b_3^2 c_1 c_2)] = 0.$$

If now $b$ and $c$ are sent into the absolute, this equation becomes

$$\sum a_1 [2c_1 a_1^2 + (c_3 - c_1) a_3 a_1 + (c_2 - c_2) a_1 a_2 - (c_2 + c_3) a_2 a_3] = 0,$$

or

$$(a_1 + a_2 + a_3) \sum 2c_1 a_1^2 - (c_2 + c_3) a_2 a_3 = 0.$$

But from the equations (5) we have

$$\Delta D I_1 + 9N = 96i \sum 2c_1 a_1^2 - (c_2 + c_3) a_2 a_3.$$

Hence, the required condition is

$$D(\Delta DI_1 + 9N) = 0.$$

The conic

$$\sum 2c_1 a_1^2 - (c_2 + c_3) a_2 a_3 = 0$$

belongs to the pencil of circles determined by the apolar- and circumcircles. Also, it evidently passes through the centroid (1, 1, 1) and orthocentre $(1/c_1, 1/c_2, 1/c_3)$, and it may be shown that the polar of the middle point of the line joining these two points is the line at infinity. Then, if the 3-point does not degenerate, $a$ must lie on the ortho-centroidal circle of the 3-line: i. e., on the circle having the join of the orthocentre and centroid as a diameter.

XV. *Let there be a collineation, having the* 3*-point as fixed triangle, which sends the* 3*-line into an inscribed* 3*-line.*

The required condition may evidently be found by applying the substitution (6) to the equation

$$D(\Delta DI_1 + 9N) = 0.$$

This gives

$$\Delta(\Delta DI_1 + 9N) = 0.$$

The condition is then, upon neglecting the cases of degenerate 3-line and 3-point, the same as that of XIV. Hence we have the theorem:

*If there exists a collineation, having fixed points at $a$, $b$ and $c$, which sends each of the points $A$, $B$ and $C$ into a point on the join of the other two, then there also exists a collineation, with fixed points at $A$, $B$ and $C$, which sends each of the points $a$, $b$ and $c$ into a point on the join of the other two.**

The vanishing of the invariant $\Delta DI_1 + 9N$ then expresses the fact that there is a mutual relation between the two triangles. The condition that the vanishing of an invariant express such a relation is that the substitution (6), applied to that invariant, shall merely reproduce it.

We have

$$\Delta' D' I_1' + \lambda N' = \frac{\Delta^2 D^2}{6}[(\lambda - 3)\Delta DI_1 + 3(9 + \lambda)N].$$

If this is to be identical to

$$k\Delta^2 D^2 [\Delta DI_1 + \lambda N],$$

we must have $\lambda$ equal to $-3$ or $9$. Then the only invariants of the pencil the vanishing of which express mutual relations are

$$\Delta DI_1 + 9N \quad \text{and} \quad \Delta DI_1 - 3N.$$

The former we have just considered, and the latter is the condition that there be a conic circumscribed to the 3-line and to the 3-point. From the substitu-

* F. MORLEY, *Projective Coördinates*, Transactions of the American Mathematical Society, vol. 4 (1903), p. 292.

tion (6), we also see that the vanishing of the invariant $I_2$ expresses a mutual relation between the two triangles. An adequate interpretation of this relation has not however presented itself.

The results of § 5 may be tabulated as follows:

The 3-point degenerates, if

$$D = 0.$$

The 3-point is apolar to the 3-line, if

$$I_1 = 0.$$

The meets of the 3-line are apolar to the joins of the 3-point, if

$$\Delta D I_1 - 9N = 0.$$

A conic, circumscribed to the 3-point and apolar to the 3-line, exists, if

$$\Delta D I_1 + 3N = 0.$$

A point conic, apolar to both the 3-point and 3-line, exists, if

$$D^2(\Delta D I_2 - 3N) = 0.$$

A conic, circumscribed to both the 3-point and 3-line, exists if

$$\Delta D I_1 - 3N = 0.$$

There exists a point, whose polar conic as to the 3-line is apolar to the 3-point, if

$$\Delta D N = 0.$$

The three polar lines, as to the 3-line, of the points of the 3-point, taken two at a time, meet in a point, if

$$\Delta N = 0.$$

There exists a collineation, having the 3-point as fixed triangle, which sends each meet of the 3-line into a point on the opposite line, if

$$\Delta D I_1 + 9N = 0.$$

The 3-line degenerates, if

$$\Delta = 0.$$

The 3-line is apolar to the 3-point, if

$$I_1 = 0.$$

The joins of the 3-point are apolar to the meets of the 3-line, if

$$\Delta D I_1 - 9N = 0.$$

A conic, inscribed to the 3-line and apolar to the 3-point, exists, if

$$\Delta D I_1 + 3N = 0.$$

A line conic, apolar to both the 3-line and 3-point, exists, if

$$\Delta^2(\Delta D I_2 - 3N) = 0.$$

A conic, inscribed to both 3-line and 3-point, exists if

$$\Delta D I_1 - 3N = 0.$$

There exists a line, whose polar conic as to the 3-point is apolar to the 3-line, if

$$\Delta D N = 0.$$

The three polar points, as to the 3-point, of the lines of the 3-line, taken two at a time, lie on a line, if

$$D N = 0.$$

There exists a collineation, having the 3-line as fixed triangle, which sends each join of the 3-point into a line through the opposite point, if

$$\Delta D I_1 + 9N = 0.$$

There exists a collineation, having the meets of the 3-line as fixed points, which sends each point of the 3-point into a point on the join of the other two, if

$$\Delta DI_1 + 9N = 0.$$

There exists a collineation, having the joins of the 3-point as fixed lines, which sends each line of the 3-line into a line through the meet of the other two, if

$$\Delta DI_1 + 9N = 0.$$

§ 6. *Two triangles apolar in both ways.*

If the two invariants, $\Delta DI_1 - 3N$ and $\Delta DI_1 + 3N$, simultaneously vanish, the two triangles formed by the 3-point and 3-line are apolar in both ways. But, if $a$ be considered as the variable point, these invariants represent a circumconic and an apolar conic of the 3-line, which meet in four points $I$, $J$, $K$, and $L$. These points pair off into two on the line $D$ and two on the line $I_1$, the former pair being of course, the points $b$ and $c$, assumed for the moment to be $I$ and $J$.

If then $a$ coincides with either $K$ or $L$, it forms with $I$ and $J$ a triangle apolar in both ways to that of the 3-line. An interchange of the pairs $I$, $J$ and $K$, $L$ merely interchanges the lines $D$ and $I_1$. Then the triangle formed by any three of the points $I$, $J$, $K$, $L$ is apolar in both ways to that of the 3-line. That is, the points $I$, $J$, $K$, $L$ form a *conjugate 4-point* of the 3-line.*

We shall now show that *the Feuerbach conic of the 3-line, $I$ and $J$ is identical with that of the 3-line, $K$ and $L$.*

Let $a$ be a point on the line $I_1$, and let $b$ and $c$ be the absolute. Then the polar of $a$ as to the 3-line is apolar to the absolute. If $a^{(1)}$ denote the point where $\alpha$ cuts the line $I_1$, then the polar conic of $a^{(1)}$ as to the 3-line breaks up into the line $\alpha$ and the perpendicular from the opposite vertex. It follows that the polar point of the line $I_1$ as to the 3-line is the orthocentre.

The Feuerbach conic of $K$, $L$ and the 3-line may be defined as the conic through $K$, $L$ and the polar points of the join of $K$ and $L$ as to the meets of the 3-line taken two at a time. This corresponds to the fact that the Feuerbach circle passes through the middle points of the sides. But, we have seen that the polars of $I_1$ as to the meets of the 3-line are the feet of the perpendiculars from the vertices of the 3-line. Then, since the Feuerbach circle also passes through these points, the two conics have five points in common, and so coincide.

Hence, the Feuerbach conic of a 3-line and any two points of a conjugate 4-point is identical with that of the 3-line and the two other points.

8. *An extension of Majcen's theorem.*†

MAJCEN shows that if we project the points of the triangle $ABC$, from an intersection of the circumcircle and apolar circle, upon the apolar circle the tri-

*CAPORALI, *Works*, p. 51.

†G. MAJCEN, Archiv der Mathematik und Physik, ser. 3, vol. 4 (1903), pp. 94, 95.

angle so formed $A^*B^*C^*$ will be circumscribed and similar to $ABC$. In this section we shall prove the following extension of this theorem:

*There exists a collineation, leaving fixed three points of a conjugate* 4*-point of a* 3*-line, which sends the* 3*-line into a circumscribed, perspective* 3*-line, the center of perspective being the remaining point of the* 4*-point, and the conic through the fixed points and circumscribed to the new* 3*-line being apolar to the original* 3*-line.*

We shall make use of the system of circular coördinates. A point of the plane will be defined by two conjugate complex quantities $x$ and $\bar{x}$, the real and imaginary parts of which are respectively the $x$ and $y$ rectangular Cartesian coördinates of the point. In order that $x$ may always define a single point, the convention is made that infinity is a point.

The bilinear substitution

$$x' = \frac{ax + b}{cx + d}$$

sends circles into circles, provided that a line be considered as a circle through the infinite point. The substitution has, in general, two fixed points. If infinity be one of these, the substitution sends circles and lines into circles and lines, and is orthomorphic. It is then identical with a collineation having fixed points at the absolute.

Let the meets of the 3-line be the points $a_1$, $a_2$, and $a_3$, and so take the axes and unit of measure that these points lie on the unit circle, and that the Euler line of the triangle $a$ is the axis of reals. Further, let $s_i$ denote the sum of the $a$'s $i$ at a time. The orthocenter and centroid are then the points $s_1$ and $s_1/3$ respectively.

Any substitution, with fixed points at infinity and $K$, is of the form

$$x' = \mu(x - K) + K. \tag{7}$$

If this is to send $a_1$ into a point on the join of $a_2$ and $a_3$, we must have

$$(1 + \lambda)[\mu(a_1 - K) + K] = a_2 + \lambda a_3$$

for a real $\lambda$. The conjugate of this equation is

$$\frac{a_2 a_3}{a_1}(1 + \lambda)[\bar{\mu}(1 - a_1\bar{K}) + a_1\bar{K}] = a_3 + \lambda a_2,$$

and hence, by addition,

$$\mu(a_1 - K)a_1 + Ka_1 + a_2a_3\bar{\mu}(1 - a_1\bar{K}) + s_3\bar{K} = a_1(a_2 + a_3).$$

Similarly we have

$$\mu(a_2 - K)a_2 + Ka_2 + a_3a_1\bar{\mu}(1 - a_2\bar{K}) + s_3\bar{K} = a_2(a_3 + a_1).$$

Hence, by subtraction,

$$\mu(a_1 + a_2 - K) + K - a_3\bar{\mu} = a_3,$$
$$\mu(a_2 + a_3 - K) + K - a_1\bar{\mu} = a_1.$$

Eliminating from the last two equations first $k$ and then $\bar{\mu}$, we have

$$\mu + \bar{\mu} + 1 = 0,$$

and

$$K = \frac{\mu s_1}{\mu - 1}. \tag{8}$$

Hence the condition upon the collineation is that the real part of $\mu$ is $-\frac{1}{2}$. The condition upon $K$ is that it must run along the ortho-centroidal circle of the triangle $a$.*

Since the intersections of the circum- and apolar circles lie on the ortho-centroidal circle, there exists a substitution, with fixed points at infinity and either of these intersection points, which sends the triangle $a$ into an inscribed triangle. The Feuerbach circle also belongs to this pencil, and therefore the points of intersection may be determined from the circum- and Feuerbach circles.

The substitution

$$x' = -\tfrac{1}{2}(x - s_1)$$

sends the points of the triangle each into the middle point of the join of the other two, and therefore sends the circumcircle into the Feuerbach circle. The map-equation of the latter is then

$$y = -\tfrac{1}{2}(t - s_1),$$

as $t$ runs along the unit circle.

If $y$ be also on the circumcircle, $|y| = 1$, and hence the points of intersection are found by solving

$$\left|\frac{t - s_1}{2}\right| = |t| = 1.$$

We have

$$\left|\frac{t - s_1}{2}\right| = \tfrac{1}{4}(t - s_1)\left(\frac{1}{t} - s_1\right) = 1.$$

Then

$$t = \frac{-2 - B \pm iA}{2s_1}$$

where

$$B = 1 - s_1 \qquad \text{and} \qquad A = \sqrt{-B(B + 8)}.\dagger$$

The points of intersection are then

* That $K$ must run along the ortho-centroidal circle as $\mu$ runs along the line $-\frac{1}{2}$ may be deduced from the results of § 5, XIV, or may be easily shown directly.

† $A$ is real or imaginary according as the points of intersection are real or imaginary.

$$(9) \qquad K=\frac{1}{4s_1}(4-B+iA), \qquad L=\frac{1}{4s_1}(4-B-iA).$$

If then in the substitution (7) we replace $\mu$ and $K$ by their values found from (8) and (9), we shall have a substitution, L, with fixed points at infinity and $K$, which sends the triangle $a$ into an inscribed triangle $a'$.

This substitution is

$$\mathrm{L}: \qquad x'=\frac{-B+iA}{2B}(x-s_1).$$

The points $L$ and $a_1$ are sent into

$$L'=\frac{-B-2+iA}{2s_1},$$

and

$$a_1'=\frac{-B+iA}{2B}(a_1-s_1).$$

Then the three point $a_1$, $a_1'$, and $L'$ lie on a line if

$$(1+\lambda)\frac{-B-2+iA}{2s_1}=\frac{-B+iA}{2B}(a_1-s_1)+\lambda a_1$$

for a real $\lambda$. The conjugate of this equation is

$$(1+\bar{\lambda})\frac{-B-2-iA}{2s_1}=\frac{-B-iA}{2B}\left(\frac{1}{a_1}-s_1\right)+\frac{\bar{\lambda}}{a_1}.$$

Eliminating $a_1$, we have

$$AB(1-B)(\lambda-\bar{\lambda})=0.$$

Then $\lambda$ is real unless $B=0$, $-8$ or $1$.

If $B=0$, the substitution sends all points into infinity.

If $B=-8$, $s_1$ is equal to $\pm 3$, and the condition that $a_1$, $a_1'$, and $L'$ lie on a line becomes

$$\pm(1+\lambda)=-\tfrac{1}{2}a_1\pm\tfrac{3}{2}+\lambda a_1,$$

which is satisfied if $\lambda=1/2$.

If $B=1$, $s_1$ is zero, and therefore $L'$ is infinite.

Then, in all cases, the points $a_1$, $a_1'$, and $L'$ lie on a line. It follows that the triangles $a$ and $a'$ are in perspective, and that the center of perspective is the point $L'$. The two triangles are also similar since infinity is a fixed point of the substitution.

The inverse substitution

$$\mathrm{L}^{-1}: \qquad x'=\frac{B+iA}{4}x+s_1,$$

sends $a$ into a circumscribed, similar and perspective triangle $a''$, the center of perspective being the point $L$. Since $L^{-1}$ sends the circumcenter, $O$, into the orthocenter, $s$, and leaves $K$ fixed, it sends the circumcircle into the apolar circle. That is, the circumcircle of $a''$ is the apolar circle of $a$. We have then proved the theorem stated at the beginning of this section.*

JOHNS HOPKINS UNIVERSITY,
*May*, 1903.

* If the conic through the fixed points and circumscribed to $a$ be apolar in both ways to that through the fixed points and apolar to $a$, the triangles $a$, $a'$, and $a''$ form a closed system. Each is circumscribed to and in perspective with the one following. These triangles form a special case of the $(3, 3)_9$ configuration $a$. See KANTOR, *Uber die Configurationen* $(3, 3)$ *mit den Indices* 8, 9, Wiener Sitzungsberichte, vol. 84, (1881), p. 918.

# ISOTHERMAL SYSTEMS OF GEODESICS*

BY

EDWARD KASNER

The object of this note is to determine the surfaces upon which there exist one or more isothermal systems composed of $\infty^1$ geodesic lines. Such a system of geodesics together with the orthogonal trajectories† may be regarded as dividing the surface into infinitesimal squares. The solution of the question leads in particular to an interesting characteristic property of the surfaces of constant curvature.

Upon any surface of revolution there is certainly one system of the kind described, namely, the meridians. The same is true for the surfaces which are applicable on a surface of revolution, i. e., for the class of surfaces whose linear element may be expressed in the form

$$ds^2 = \lambda(x)(dx^2 + dy^2). \tag{1}$$

We now prove the converse result:

*If a surface possesses an isothermal system of geodesics, then it is applicable on a surface of revolution, the geodesics corresponding to the meridians.*

For assume the system of geodesics to be the parameter lines $y = \text{const.}$, and the orthogonal trajectories to be the parameter lines $x = \text{const.}$‡ Then the linear element may be written

$$ds^2 = E(x, y)(dx^2 + dy^2).$$

Since the differential equation of the geodesics,

$$y'' = \frac{\partial E}{\partial x}(y' - y'^3) + \frac{\partial E}{\partial y}(y'^2 - 1),$$

is to be satisfied by $y = \text{const.}$, i. e., by $y' = 0$, it follows that $\partial E/\partial y = 0$, and hence that $ds^2$ is of the form (1) which is characteristic of the class of surfaces described in the theorem.

* Presented to the Society at the Boston meeting, August 31, 1903. Received for publication November 4, 1903.

† These form a system of (geodesically) parallel curves. The only surfaces which can be divided into infinitesimal squares by two systems of geodesics are the developable surfaces.

‡ This choice of parameter lines is legitimate, unless the system of geodesics in question is one of the systems of minimal lines. This case however is of no interest and is excluded from the following discussion.

We proceed now to examine whether more than one system of isothermal geodesics can exist upon a surface. It will be convenient to transform the element (1) by introducing the parameters $u, v$ of the minimal lines. The new form may be written

(1′) $$ds^2 = 2e^{f(u+v)}\,du\,dv.$$

The equation of the geodesics is now

(2) $$v'' = g(u+v)(v'^2 - v'),$$

where the function $g$ is the negative derivative of the function $f$,

(2′) $$g(t) = -f'(t).$$

One solution of (2) is evidently $v' = 1$ or

(3) $$u - v = \text{const.},$$

which corresponds to the meridians in the case of a surface of revolution.

Every isothermal system may be assumed in the form

(4) $$U_1 - V_1 = \text{const.},$$

where $U_1$ is a function of $u$ alone, and $V_1$ is a function of $v$ alone. If this is to represent geodesics, then

$$V_1'^2 U_1''' - U_1'^2 V_1''' = g U_1' V_1' (U_1' - V_1').$$

Dividing* both sides by $U_1'^2 V_1'^2$, this may be written

(5) $$\frac{U' - V'}{U - V} = g(u+v),$$

where $U$ and $V$ are defined

(6) $$U = \frac{1}{U_1'}, \qquad V = \frac{1}{V_1'}.$$

Since the partial derivatives $g_u$ and $g_v$ are equal, we find from (5)

(7) $$UV''' - U''' V = U'^2 - UU'' - V'^2 + VV''.$$

Finally, differentiating both sides with respect to $u$ and $v$,

(8) $$U' V'''' - V' U'''' = 0.$$

It will be convenient to consider first the case where either $U'$ or $V'$ vanishes. If for example $V' = 0$, then $V$ is a constant. Hence from (5), $U$ is also a constant. If $U$ and $V$ are equal to the same constant, system (4) is identical with system (3), so that no second system of the required kind is obtained. If $U$ and $V$ are different constants, then $g = 0$; this will be considered later.

*This is allowable, since neither $U_1'$ nor $V_1'$ can vanish unless (4) represents merely one of the minimal systems.

We may assume now that neither $U'$ nor $V'$ vanishes, and rewrite (8) in the form

$$\frac{U'''}{U'} = \frac{V'''}{V'} = c^2. \tag{9}$$

*Discussion of* (9) *for* $c = 0$.

Here we have $U''' = V''' = 0$, and hence

$$U = au^2 + a_1 u + a_2, \qquad V = bv^2 + b_1 v + b_2. \tag{10}$$

Substituting these functions in (7), we find that the coefficients must fulfill the conditions

$$a = b, \qquad 4a(a_2 - b_2) = a_1^2 - b_1^2. \tag{11}$$

Suppose first that $a \neq 0$. Then

$$a_2 - b_2 = \frac{a_1^2 - b_1^2}{4a}.$$

Hence from (5), after some calculation,

$$\frac{1}{g} = \frac{u+v}{2} + \frac{a_1 + b_1}{4a}.$$

Suppose next that $a = 0$. Then from (11) either $a_1 - b_1 = 0$ or else $a_1 + b_1 = 0$. The first relation gives

$$U = a_1 u + a_2, \qquad V = a_1 v + b_2; \qquad g = 0.$$

The second relation, on the other hand, gives

$$U = a_1 u + a_2, \qquad V = -a_1 v + b_2; \qquad \frac{1}{g} = \frac{u+v}{2} + \frac{a_2 - b_2}{2a_1}.$$

Finally if $a_1 = 0$, $U$ and $V$ reduce to constants, a case already considered.

*Discussion of* (9) *for* $c \neq 0$.

The integration of the system (9) gives now

$$U = Ae^{cu} + A_1 e^{-cu} + A_2, \quad V = Be^{cv} + B_1 e^{-cv} + B_2. \tag{12}$$

From (7) we obtain the following conditions on the constants.

$$A_2 = B_2, \quad AA_1 - BB_1 = 0. \tag{13}$$

Suppose first that neither $A$ nor $B$ vanishes. Then introducing a constant $m = A_1/B = B_1/A$, we find

$$U = Ae^{cu} + mBe^{-cu} + A_2, \quad V = Be^{cv} + mAe^{-cv} + A_2; \tag{14}$$

$$g = c\,\frac{e^{c(u+v)} + m}{e^{c(u+v)} - m}.$$

In the second place suppose that one of the constants $A$, $B$ vanishes while the other does not. Assuming for example $A = 0$, $B \neq 0$, it follows from (13) that $B_1 = 0$. Hence

$$U = A_1 e^{-cu} + A_2, \qquad V = Be^{cv} + A_2; \qquad g = c\,\frac{Be^{c(u+v)} + A_1}{Be^{c(u+v)} - A_1}.$$

Finally the supposition that both $A$ and $B$ vanish gives

$$U = A_1 e^{-cu} + A_2, \qquad V = B_1 e^{-cv} + A_2; \qquad g = -c.$$

Examining all the values of the function $g$ which present themselves, we find that they are included in the three types

$$\alpha, \quad \frac{2}{t+\beta}, \quad c\,\frac{e^{ct}+\gamma}{e^{ct}-\gamma}, \tag{15}$$

where $t$ represents the argument $u + v$, and $\alpha$, $\beta$, $\gamma$ and $c$ are constants. From (2′) the corresponding types of the function $f$ are

$$-\alpha t + \alpha_1, \qquad \log\frac{\beta_1}{(t+\beta)^2}, \qquad \log\frac{\gamma_1 e^{ct}}{(e^{ct}-\gamma)^2}. \tag{16}$$

Calculating now the Gaussian curvature of the element (1′) by means of the formula * $K = -e^{-f}f''$, we find, corresponding to the forms (16), that $K$ reduces to $0$, $2/\beta_1$, $2c^2\gamma/\gamma_1$, respectively. Thus in every case $K$ is constant, and we reach the following conclusion:

*The only surfaces upon which more than one isothermal system of geodesics can exist are the surfaces of constant (Gaussian) curvature.* †

Conversely, upon any surface of constant curvature there actually exist more than one, indeed a double infinity of systems of the kind described. This is easily shown directly from the above formulas. It is sufficient to introduce the values of $g$ given in (15) into the relation (5), thus determining $U$, $V$ and hence the required systems (4).

* This may be derived from the general formula for $K$ given on page 68 of BIANCHI, *Differentialgeometrie*, German translation of LUKAT.

† [Added December 5]. So far as this theorem is concerned, the analysis on the last two pages is obviated by the following simple proof kindly suggested by Dr. L. P. EISENHART. It has been seen that when a surface possesses an isothermal system of geodesics its element may be reduced to the form (1). For this $K$ is a function of $x$ alone, so that it is constant along the orthogonal trajectories of the geodesics. Now if a second system of the required kind exists, $K$ would also be constant along its orthogonal trajectories. It follows that unless $K$ is constant over the entire surface, the two systems must coincide.

The above analysis yields the explicit equations of the $\infty^2$ systems of geodesics, and proves at the same time that the element of a surface of constant curvature referred to any isothermal system of geodesics and the orthogonal trajectories takes the form

$$ds^2 = 2e^{f(x)}(dx^2 + dy^2),$$

where $f$ is one of the types (16).

The result however may be obtained more simply by making use of the known theory of surfaces of constant curvature.* Such a surface can be built conformally on the plane in such a manner that its geodesics are pictured by a linear two-parameter system of circles, namely, by all the circles orthogonal to a base circle of real, imaginary or zero radius according as $K$ is positive, negative or zero.† The problem of finding isothermal systems of geodesics on the surface is thus reduced to the problem of finding isothermal systems of circles contained in the two-parameter system. It is known however that the only isothermal systems of circles are the pencils, i. e., the circles passing through a pair of real or imaginary points. Such a pencil is contained in the two parameter systems only if the two points are inverse with respect to the base circle. The number of pencils is therefore $\infty^2$, so that

*Upon any surface of constant curvature there exist $\infty^2$ isothermal systems of geodesics.*

For the plane these are of course the pencils of straight lines, including the systems of parallel lines.‡ For the sphere the required systems are the great circles through a pair of antipodal points. Finally for the pseudosphere the required systems consist of the geodesics passing through any point, including as a limiting case the system of meridians, where the point may be conceived of as at infinity.

The investigation leads then to the following *classification of surfaces according to the number of isothermal systems of geodesics existing upon them:*

1°. *Surfaces applicable on no surface of revolution; no system.* 2°. *Surfaces applicable on a surface of revolution, but of variable curvature; one system.* 3°. *Surfaces of constant curvature; $\infty^2$ systems.*

The problem which has been solved is equivalent to the determination of isothermal systems consisting of (geodesically) parallel curves. For the orthogonal trajectories of such a system form an isothermal system of geodesics, and vice versa. On the general surface of revolution the only isothermal system of parallels is composed of the circles of latitude. On a surface of constant curvature there are $\infty^2$ systems, namely, the geodesic circles described about any point of the surface as center, including the limiting cases where the center may be regarded as at infinity.

COLUMBIA UNIVERSITY,
*October*, 1903.

* The forms derived by DARBOUX (*Leçons*, vol. 3, p. 219) show that the geodesics through any point constitute an isothermal system; but the converse result does not appear to have been noticed.

† For references see Transactions, vol. 4 (1903), p. 150, note.

‡ Cf. B. O. PEIRCE, *On families of curves which are the lines of certain plane vectors either solenoidal or lamellar*, Proceedings of the American Academy of Arts and Sciences, vol. 38 (1903), p. 673.

# ZUR GRUPPENTHEORIE MIT ANWENDUNGEN AUF DIE THEORIE DER LINEAREN HOMOGENEN DIFFERENTIALGLEICHUNGEN*

VON

ALFRED LOEWY

Es seien $r$ lineare homogene Substitutionen $A_1, A_2, \cdots, A_r$ der Grade $n_1, n_2, \cdots, n_r$ gegeben:

$$(A_1) \qquad x_{i_1}^{(1)} = \sum_{s_1=1}^{s_1=n_1} a_{i_1 s_1}^{(1)} \xi_{s_1}^{(1)} \qquad (i_1 = 1, 2, \cdots, n_1),$$

$$(A_2) \qquad x_{i_2}^{(2)} = \sum_{s_2=1}^{s_2=n_2} a_{i_2 s_2}^{(2)} \xi_{s_2}^{(2)} \qquad (i_2 = 1, 2, \cdots, n_2),$$

$$\vdots$$

$$(A_r) \qquad x_{i_r}^{(r)} = \sum_{s_r=1}^{s_r=n_r} a_{i_r s_r}^{(r)} \xi_{s_r}^{(r)} \qquad (i_r = 1, 2, \cdots, n_r).$$

Durch Multiplication erhält man:

$$x_{i_1}^{(1)} \cdot x_{i_2}^{(2)} \cdots x_{i_r}^{(r)} = \sum_{k_1 k_2 \cdots k_r} a_{i_1 k_1}^{(1)} a_{i_2 k_2}^{(2)} \cdots a_{i_r k_r}^{(r)} \xi_{k_1}^{(1)} \xi_{k_2}^{(2)} \cdots \xi_{k_r}^{(r)};$$

linker Hand ist dem Index $i_1$ jede der Zahlen $1, 2, \cdots, n_1$, dem Index $i_2$ jede der Zahlen $1, 2, \cdots, n_2$, u. s. w., dem Index $i_r$ jede der Zahlen $1, 2, \cdots n_r$ beizulegen. Ebenso ist in der Summe rechts für $k_1$ jede der Zahlen $1, 2, \cdots, n_1$, für $k_2$ jede der Zahlen $1, 2, \cdots, n_2$, u. s. w., für $k_r$ jede der Zahlen $1, 2, \cdots, n_r$ zu setzen. Die $n_1 n_2 \cdots n_r$ Producte $x_{i_1}^{(1)} \cdot x_{i_2}^{(2)} \cdots x_{i_r}^{(r)}$ stellen sich offenbar als lineare homogene Functionen der $n_1 n_2 \cdots n_r$ Producte $\xi_{i_1}^{(1)} \cdot \xi_{i_2}^{(2)} \cdots \xi_{i_r}^{(r)}$ dar. Die so entstandene Transformation heisst nach Herrn A. HURWITZ† die Producttransformation von $A_1, A_2, \cdots, A_r$, und wird mit $A_1 \times A_2 \times \cdots \times A_r$ bezeichnet.

Für die Producttransformationen gelten folgende leicht beweisbare Sätze. Bezüglich der Herleitung vgl. man HURWITZ a. a. O. oder auch C. STEPANOS,‡ der über diesen Gegenstand eingehende Untersuchungen angestellt hat:

* Presented to the Society at the Boston meeting, August 31—September 1, 1903. Received for publication July 1, 1903.

†A. HURWITZ, *Zur Invariantentheorie*, Mathematische Annalen, 45 (1894), S. 388. Die zu $A_1 \times A_2$ gehörige Determinante ist von KRONECKER in seinen Vorlesungen benützt worden. Vgl. NETTO in der Encyklopädie der math. Wiss., Bd. I, S. 40.

‡C. STEPHANOS, *Sur une extension du calcul des substitutions linéaires*, Journal de mathématiques, Ser. 5, Bd. 6 (1900), S. 82. STEPHANOS nennt die fragliche Operation conjonction.

Sind $B_1, B_2, \cdots B_r$ $r$ weitere lineare homogene Transformationen, aus denen die Producttransformation $B_1 \times B_2 \cdots \times B_r$ gebildet sei, so gilt die Relation:

$$(I) \qquad (A_1 \times A_2 \times \cdots \times A_r)\cdot(B_1 \times B_2 \times \cdots \times B_r) = (A_1 B_1 \times A_2 B_2 \times \cdots \times A_r B_r).$$

Die Producttransformation der inversen Transformationen von $A_1, A_2, \cdots, A_r$ ist die inverse Transformation der Producttransformation von $A_1, A_2, \cdots, A_r$, folglich ist:

$$(II) \qquad (A_1 \times A_2 \times \cdots \times A_r)^{-1} = A_1^{-1} \times A_2^{-1} \times \cdots \times A_r^{-1}.$$

Durchläuft $A_1$ sämtliche Elemente eine Gruppe linearer homogener Substitutionen, ebenso $A_2$ die sämtlichen Elemente einer Gruppe linearer homogener Substitutionen, u. s. w., schliesslich $A_r$ die sämtlichen Elemente einer Gruppe linearer homogener Substitutionen, so bildet die Gesamtheit von linearen homogenen Substitutionen $A_1 \times A_2 \times \cdots \times A_r$, welche auf diese Weise entstehen, auch eine Gruppe. *

Auf das gruppentheoretische Princip, welches in der Producttransformation liegt, hat übrigens schon Herr C. JORDAN in seinem *Traité des substitutions et des équations algébriques* † hingewiesen. Man kann nämlich auch auf folgende Weise zur Producttransformation gelangen: Es sei

$$\sum_{i_1 i_2 \cdots i_r} c_{i_1 i_2 \cdots i_r} x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_r}^{(r)}$$

eine Form, die in den $r$ Variablenreihen $x_{i_1}^{(1)}$ $(i_1 = 1, 2, \cdots, n_1)$, $x_{i_2}^{(2)}$ $(i_2 = 1, 2, \cdots, n_2)$, $\cdots$, $x_{i_r}^{(r)}$ $(i_r = 1, 2, \cdots, n_r)$ linear und homogen ist. Transformirt man die angegebene Form durch die Substitutionen $A_1, A_2, \cdots, A_r$ in die neue $r$ fach lineare Form:

$$\sum_{i_1 i_2 \cdots i_r} c'_{i_1 i_2 \cdots i_r} \xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \cdots \xi_{i_r}^{(r)},$$

so wird:

$$c'_{k_1 k_2 \cdots k_r} = \sum_{i_1 i_2 \cdots i_r} a_{i_1 k_1}^{(1)} a_{i_2 k_2}^{(2)} \cdots a_{i_r k_r}^{(r)} c_{i_1 i_2 \cdots i_r},$$

d. h. aber die $n_1 n_2 \cdots n_r$ Coefficienten $c'_{k_1 k_2 \cdots k_r}$ werden durch $A'_1 \times A'_2 \cdots \times A'_r$ transformirt; dabei bedeuten $A'_1, A'_2 \cdots, A'_r$ die transponirten Substitutionen zu $A_1, A_2, \cdots, A_r$. ‡

* Die Producttransformation wird auch in der FROBENIUS'schen Theorie der Composition der Charaktere einer endlichen Gruppe verwandt. Vgl. FROBENIUS, *Über die Composition der Charaktere einer Gruppe*, Sitzungsber. der Kgl. preussischen Akademie der Wiss. (1899), S. 330; W. BURNSIDE, *On the composition of the group-characteristics*, Proceedings of the London Mathematical Society, vol. 34 (1901), p. 41.

† a. a. O., S. 221.

‡ Vgl. HURWITZ, a. a. O., S. 394. Siehe auch FRANKLIN, *Note on induced linear substitutions*, American Journal of Mathematics, vol. 16 (1894), p. 206.

Ist $A$ eine gegebene lineare homogene Substitution und wird

$$A_1 = A_2 = \cdots = A_r = A$$

gesetzt, also ist $A \times A \times A \times \cdots \times A$ genau $r$ mal zu nehmen, so bezeichne ich diese Operation mit Herrn ISSAI SCHUR * als $\Pi_r A$; da für $r = 1$ $\Pi_1 A = A$ ist, nehmen wir $r > 1$ an.

Ist $A$ eine gegebene Substitution:

$$x_i = \sum_{s=1}^{s=n} a_{is}\xi_s \qquad (i = 1, 2, \cdots, n), \tag{1}$$

so denke man sich, um $\Pi_r A$ zu erhalten, die Substitution $A$ in den zweimal $r$ Variablensystemen $x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(r)}$ bez. $\xi_i^{(1)}, \xi_i^{(2)}, \cdots, \xi_i^{(r)}$ geschrieben:

$$x_i^{(1)} = \sum_{s=1}^{s=n} a_{is}\xi_s^{(1)},\ x_i^{(2)} = \sum_{s=1}^{s=n} a_{is}\xi_s^{(2)}, \cdots, x_i^{(r)} = \sum_{s=1}^{s=n} a_{is}\xi_s^{(r)} \qquad (i = 1, 2, \cdots, n). \tag{1'}$$

Dann wird:

$$x_{i_1}^{(1)} \cdot x_{i_2}^{(2)} \cdots x_{i_r}^{(r)} = \sum_{k_1 k_2 \ldots k_r} a_{i_1 k_1} \cdot a_{i_2 k_2} \cdots a_{i_r k_r} \xi_{k_1}^{(1)} \xi_{k_2}^{(2)} \cdots \xi_{k_r}^{(r)}; \tag{2}$$

für die Indices $i_1, i_2, \cdots, i_r$ linker Hand ist jede der Zahlen $1, 2, \cdots, n$ zu setzen; ebenso ist in der Summe rechter Hand für $k_1, k_2, \cdots, k_r$ jede der Zahlen $1, 2, \cdots, n$ zu nehmen. Wir bezeichnen die $n^r$ verschiedenen Producte $x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_r}^{(r)}$ mit $X_{i_1 i_2 \ldots i_r}$ und analog die $n^r$ Producte $\xi_{i_1}^{(1)} \xi_{i_2}^{(2)} \cdots \xi_{i_r}^{(r)}$ mit $\Xi_{i_1 i_2 \ldots i_r}$; dann haben wir die auf $n^r$ Variable sich beziehende Transformation $\Pi_r A$. Sie lautet:

$$X_{i_1 i_2 \ldots i_r} = \sum_{k_1 k_2 \ldots k_r} a_{i_1 k_1} \cdot a_{i_2 k_2} \cdots a_{i_r k_r} \Xi_{k_1 k_2 \ldots k_r}. \tag{3}$$

Werden in (2) die oberen Indices $1, 2, \cdots, r$ bei den $x$ und $\xi$ fortgelassen, so entstehen die Producte $x_{i_1} x_{i_2} \cdots x_{i_r}$; ihre Anzahl ist bekanntlich $\binom{n+r-1}{r}$, nämlich gleich der Zahl, die angiebt, auf wieviele Arten man $n$ Zahlen zu $r$ mit Wiederholung combiniren kann. Diese $\binom{n+r-1}{r}$ Producte $x_{i_1} x_{i_2} \ldots x_{i_r}$ transformiren sich linear in die $\binom{n+r-1}{r}$ Producte $\xi_{i_1} \xi_{i_2} \cdots \xi_{i_r}$.

Die zugehörige lineare Transformation heisse nach Herrn HURWITZ' † Vorgang die $r$te *Potenztransformation* von $A$ und werde mit $P_r A$ bezeichnet. Für sie gelten dann die Sätze: Sind $A$ und $B$ zwei Substitutionen, so ist $P_r A \cdot P_r B = P_r (A B)$, ‡ ferner ist $P_r (A^{-1}) = (P_r A)^{-1}$.

* ISSAI SCHUR, *Über eine Klasse von Matricen, die sich einer gegebenen Matrix zuordnen lassen*, Berliner Dissertation, 1901, S. 20. Mit $\Pi_r A$ beschäftigt sich auch W. BURNSIDE's Arbeit, *On the characteristic equations of certain linear substitutions*, Quarterly Journal of pure and applied Mathematics, vol. 32 (1901), p. 80.

† HURWITZ, a. a. O., S. 390.

‡ Die Litteratur über die Potenztransformationen findet man bei I. SCHUR, a. a. O., S. 17. Hierzu ist noch beizufügen: W. BURNSIDE, a. a. O.; G. RADOS, *Inducirte lineare Substitutionen*, Math. u. naturw. Berichte aus Ungarn, Bd. 16 (1898), S. 241.

$P_r A$ und $\Pi_r A$ sind von einander verschieden, ausgenommen für $n=1$, d. h. wenn $A$ eine Substitution in einer einzigen Variablen ist. *Dieser triviale Fall $n=1$ soll im Folgenden ausgeschlossen sein.*

Sind $l_1, l_2, \cdots, l_r$ irgend $r$ unter einander verschiedene der $n$ Zahlen $1, 2, \cdots, n$, die in der Reihenfolge $l_1 < l_2 \cdots < l_r$ gewählt sein sollen, so bilde man für jede mögliche Wahl der Zahlen $l_1, l_2, \cdots, l_r$ die Determinante:

$$(4) \qquad T_{l_1 l_2 \ldots l_r} = \begin{vmatrix} x_{l_1}^{(1)} & x_{l_2}^{(1)} & \cdots & x_{l_r}^{(1)} \\ x_{l_1}^{(2)} & x_{l_2}^{(2)} & \cdots & x_{l_r}^{(2)} \\ \vdots & & & \\ x_{l_1}^{(r)} & x_{l_2}^{(r)} & \cdots & x_{l_r}^{(r)} \end{vmatrix}.$$

Solcher Determinanten giebt es $\binom{n}{r}$, nämlich soviele, als man $n$ Zahlen zu $r$ verschiedenen combiniren kann. $r$ muss $\leqq n$ sein. Transformirt man die $r$ Variablensysteme $x_i^{(1)}, x_i^{(1)}, \cdots, x_i^{(r)}$ cogredient durch die Substitution $A$ nach (1′), so erfahren die $\binom{n}{r}$ Determinanten $T_{l_1 l_2 \ldots l_r}$ auch eine lineare Substitution. Diese nennt Herr Hurwitz* die $r$te *Determinantentransformation*; sie werde nach ihm mit $C_r A$ bezeichnet. Da $C_1 A = A$ ist, so soll bei $C_r A$, für welches $r \leqq n$ ist, auch $r=1$ ausgeschlossen werden.

Durchläuft $A$ die Elemente einer *Gruppe $G$ linearer homogener Substitutionen*, und bildet man zu jeder Substitution $A$ von $G$ die Substitution $\Pi_r A$, so erhält man eine zu $G$ isomorphe Gruppe, die mit $\Pi_r G$ bezeichnet sei.

Auf gleiche Art und Weise können aus $G$ eine Gruppe $P_r G$ und eine Gruppe $C_r G$ hergeleitet werden, indem man zu jeder Substitution $A$ von $G$ die Substitutionen $P_r A$ bez. $C_r A$ bildet. $C_r G$ existirt nur für $1 < r \leqq n$, wenn $G$ eine Gruppe in $n$ Variablen ist.

$H_r A$ soll diejenige Substitution sein, welche durch die zerlegbare Matrix:

$$(5) \qquad \begin{matrix} P_r A & 0 \\ 0 & C_r A \end{matrix}$$

bestimmt ist; für $r \leqq n+1$ ist unter $H_r A$ einfach $P_r A$ zu verstehen. $H_1 A$, das im Folgenden auch verwandt werden wird, soll $A$ sein. Ordnet man einer jeden Substitution $A$ einer Gruppe $G$ linearer homogener Substitutionen die Substitution $H_r A$ zu, so bildet die Gesamtheit der Substitutionen $H_r A$, welche

* HURWITZ, a. a. O., S. 392. Die Litteratur über die Determinantentransformation ist sehr ausgedehnt. Bezüglich der Litteratur sei verwiesen auf VAHLEN, Encyklopädie der math. Wiss., Bd. I, S. 594 und LUDW. SCHLESINGER, *Handbuch der Theorie der linearen Differentialgleichungen*, $II_1$, S. VIII (Zehnter Abschnitt, zweites Capitel). Hierzu ist die von WEIERSTRASS stammende in BALTZER'S *Determinantentheorie* (4. Aufl. 1875), S. 55 übergegangene Bemerkung beizufügen. Vgl. auch L. E. DICKSON, *Linear groups with an exposition of the Galois Field theory*, S. 145 ff. sowie A. LOEWY, *Über Differentialgleichungen, die mit ihren adjungirten zu derselben Art gehören*, Sitzungsber. der math. phys. Klasse der kgl. bayer. Akademie der Wiss. (1902), S. 5.

allen Substitutionen $A$ von $G$ entsprechen, eine zu $G$ isomorphe Gruppe; diese sei mit $H_r G$ oder auch mit

$$(5') \qquad \begin{matrix} P_r G & 0 \\ 0 & C_r G \end{matrix}$$

bezeichnet.

Im § 1 beweise ich:

*Ist $A$ eine lineare homogene Substitution, so werden bei gegebenem $r(r>1)$ gleichzeitig alle Substitutionen $\Pi_r A$, wobei für $A$ jede Substitution der gleichen Variablenzahl gesetzt werden darf, durch eine und dieselbe Substitution $S$ von nicht verschwindender Determinante in die Form:*

$$(6) \qquad S\Pi_r A S^{-1} = \begin{matrix} H_r A & 0 \\ A_{21} & A_{22} \end{matrix}$$

*übergeführt; $A_{21}$ und $A_{22}$ bedeuten dabei Matrices von $n^r - \binom{n}{r} - \binom{n+r-1}{r}$ Zeilen. Die besondere Einführung der Potenz- und Determinantentransformation ist also entbehrlich. Potenz- und Determinantentransformation treten auch bei der Producttransformation auf.*

*Ist $G$ irgend eine Gruppe einer endlichen oder unendlichen Anzahl linearer homogener Substitutionen, so ist die zugehörige Gruppe $\Pi_r G$, da alle ihre Substitutionen gleichzeitig durch eine und dieselbe Substitution in die Form (6) gebracht werden können, reducibel, und $\Pi_r G$ kann so transformirt werden dass als einer der Bestandteile oder eine der Teilgruppen die Gruppe $H_r G$ erscheint.*

Im § 2 leite ich den Satz her, dass, wenn $r = r_1 + r_2 \cdots + r_f$ irgend eine Zerlegung der ganzen positiven Zahl $r$ in ganzzahlige positive Summanden bedeutet, die Gruppe $\Pi_r G$ auch so transformirt werden kann, dass die Gruppe $H_{r_1} G \times H_{r_2} G \times \cdots \times H_{r_f} G$ als ein Bestandteil oder eine Teilgruppe von $\Pi_r G$ erscheint; alle Substitutionen $\Pi_r A$ von $\Pi_r G$ lassen sich nämlich gleichzeitig durch dieselbe Matrix von nicht verschwindender Determinante in die Form:

$$(7) \qquad \begin{matrix} H_{r_1} A \times H_{r_2} A \times \cdots \times H_{r_f} A & 0 \\ L_{21} & L_{22} \end{matrix}$$

bringen.* Aus diesem Satze kann ferner geschlossen werden, das $\Pi_r G$ auch stets einer zerlegbaren oder zerfallenden Gruppe ähnlich ist.

Im § 3 werden die vorstehenden Resultate auf die *Theorie der linearen homogenen Differentialgleichungen* angewandt; hierdurch gelangt man zu *Resolventen einer linearen homogenen Differentialgleichung, die ebenfalls linear und homogen* sind. Die Substitutionen $C_r A$ ergeben Differentialglei-

* Man beachte, dass im allgemeinen $H_r A$ von $H_{r_1} A \times H_{r_2} A \times \cdots \times H_{r_f} A$ verschieden ist; denn beide Substitutionen haben im allgemeinen verschiedene Variablenzahl.

chungen, welche die in Herrn LUDW. SCHLESINGER's *Handbuch* * nach Herrn FORSYTH † sogenannten associerten Differentialgleichungen einer linearen homogenen Differentialgleichung umfassen. Neben diese linearen homogenen Differentialgleichungen treten noch weitere Gattungen linearer homogener Differentialgleichungen, die zu einer vorgelegten linearen homogenen Differentialgleichung gehören und den Operationen $\Pi_r A$ bez. $P_r A$ entsprechen.

## § 1.

Die Substitution $\Pi_r A$, mit der wir uns beschäftigen wollen, lautet nach (3):

$$X_{i_1 i_2 \dots i_r} = \sum_{k_1 k_2 \dots k_r} a_{i_1 k_1} \cdot a_{i_2 k_2} \cdots a_{i_r k_r} \Xi_{k_1 k_2 \dots k_r}. \tag{3}$$

Es möge $C_{i_1 i_2 \dots i_r}$ die Anzahl der verschiedenen Permutationen der $r$ Zahlen $i_1, i_2, \dots, i_r$ bedeuten. Sind unter den $r$ Zahlen $i_1, i_2, \dots i_r$, die sämtlich den $n$ Zahlen $1, 2, \dots, n$ entstammen, $r_1$ unter einander gleich einer und derselben Zahl, $r_2$ gleich einer zweiten Zahl, u. s. w., so ist

$$C_{i_1 i_2 \dots i_r} = \frac{r!}{r_1! r_2! \dots}.$$

Wir setzen:

$$Y_{i_1 i_2 \dots i_r} = \frac{\sum' X_{i_1 i_2 \dots i_r}}{C_{i_1 i_2 \dots i_r}}. \tag{8}$$

Die Summe rechts soll alle Summanden umfassen, die erhalten werden, indem man die Zahlen $i_1, i_2, \dots, i_r$ alle ihre verschiedenen Permutationen annehmen lässt.

$$\sum'_{i_1 i_2 \dots i_r} X_{i_1 i_2 \dots i_r}$$

enthält mithin $C_{i_1 i_2 \dots i_r}$ Glieder. Unter einander verschiedene Grössen $Y_{i_1 i_2 \dots i_r}$ giebt es soviele, wie $n$ Zahlen zu $r$, wobei auch die $r$ Elemente mehrfach auftreten, ohne Rücksichtnahme auf die Reihenfolge, verbunden werden können (Combinationen mit Wiederholung), also

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}.$$

Analog zu (8) setzen wir:

$$\Upsilon_{i_1 i_2 \dots i_r} = \frac{\sum' \Xi_{i_1 i_2 \dots i_r}}{C_{i_1 i_2 \dots i_r}}. \tag{8'}$$

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* LUDW. SCHLESINGER, *Handbuch der Theorie der linearen Differentialgleichungen* (1897), Leipzig, $II_1$, 127.

† Royal Society Transactions, vol. 179 (1888), p. 424.

Aus (3) und (8) folgt:

$$(9) \qquad Y_{i_1 i_2 \dots i_r} = \frac{\sum'_{i_1 i_2 \dots i_r} \sum_{k_1 k_2 \dots k_r} a_{i_1 k_1} \cdot a_{i_2 k_2} \cdots a_{i_r k_r} \Xi_{k_1 k_2 \dots k_r}}{C_{i_1 i_2 \dots i_r}}.$$

In (9) hat die Variable $\Xi_{k_1 k_2 \dots k_r}$ den Factor:

$$(10) \qquad \frac{\sum'_{i_1 i_2 \dots i_r} a_{i_1 k_1} a_{i_2 k_2} \cdots a_{i_r k_r}}{C_{i_1 i_2 \dots i_r}};$$

dabei ist $i_1, i_2, \cdots, i_r$ durch die Zeile in (9) bestimmt, aus der $\Xi_{k_1 k_2 \dots k_r}$ genommen wurde. Bedeutet $k'_1, k'_2, \cdots, k'_r$ eine Permutation der Zahlen $k_1, k_2, \cdots, k_r$, so hat die aus derselben Zeile in (9) wie $\Xi_{k_1 k_2 \dots k_r}$ entnommene Variable $\Xi_{k'_1 k'_2 \dots k'_r}$ den Factor:

$$(11) \qquad \frac{\sum'_{i_1 i_2 \dots i_r} a_{i_1 k'_1} \cdot a_{i_2 k'_2} \cdots a_{i_r k'_r}}{C_{i_1 i_2 \dots i_r}}.$$

Die Bezeichnung $\sum'_{i_1 i_2 \dots i_r}$ ist in (10) und (11) so zu verstehen, dass die $r$ Zahlen $i_1, i_2, \cdots, i_r$ alle ihre $C_{i_1 i_2 \dots i_r}$ verschiedenen Permutationen annehmen sollen. Da $k'_1, k'_2, \cdots, k'_r$ nur eine Permutation der Zahlen $k_1, k_2, \cdots, k_r$ vorstellt, so kann der Zähler

$$\sum'_{i_1 i_2 \dots i_r} a_{i_1 k'_1} \cdot a_{i_2 k'_2} \cdots a_{i_r k'_r}$$

in (11) so angeordnet werden, dass in jedem Summanden der zweite Index des ersten Factors $k_1$, des zweiten Factors $k_2$, u. s. w., des letzten Factors schliesslich $k_r$ lautet. Da sich in (10) und (11) $\sum'_{i_1 i_2 \dots i_r}$ nur auf $i_1, i_2, \cdots, i_r$ bezieht und alle verschiedenen Permutationen der $r$ Zahlen $i_1, i_2, \cdots, i_r$ zu benützen sind, so ersieht man, dass (10) und (11) gleich sind. In (9) haben daher alle $\Xi$, die aus einem $\Xi_{k_1 k_2 \dots k_r}$ durch Permutation der Indices $k_1, k_2, \cdots, k_r$ hervorgehen und derselben Zeile in (9) angehören, denselben Factor, nämlich:

$$\frac{\sum'_{i_1 i_2 \dots i_r} a_{i_1 k_1} \cdot a_{i_2 k_2} \cdots a_{i_r k_r}}{C_{i_1 i_2 \dots i_r}};$$

wir bezeichnen diesen Factor mit

$$(12) \qquad b^{k_1 k_2 \dots k_r}_{i_1 i_2 \dots i_r}.$$

Bei fest gegebenem $i_1, i_2, \cdots, i_r$ ändert sich $b^{k_1 k_2 \dots k_r}_{i_1 i_2 \dots i_r}$ nicht, wenn die oberen Indices $k_1, k_2, \cdots, k_r$ permutirt werden. Unter Beachtung von (12) wird (9) übergehen in:

$$(13) \qquad Y_{i_1 i_2 \dots i_r} = \sum_{k_1 k_2 \dots k_r} b^{k_1 k_2 \dots k_r}_{i_1 i_2 \dots i_r} \Xi_{k_1 k_2 \dots k_r}.$$

Betrachtet man $\Xi_{k_1 k_2 \dots k_r}$ und die durch Permutation der $k_1, k_2, \cdots, k_r$ hieraus hervorgehenden Grössen, so bezeichnen wir die Summe aus diesen $C_{k_1 k_2 \dots k_r}$ Summanden mit

$$\sum_{k_1 k_2 \dots k_r}' \Xi_{k_1 k_2 \dots k_r}.$$

Die Factor dieses Aggregats

$$\sum_{k_1 k_2 \dots k_r}' \Xi_{k_1 k_2 \dots k_r}$$

lautet in (13) wegen der Eigenschaft von $b_{i_1 i_2 \dots i_r}^{k_1 k_2 \dots k_r}$ sich bei den Permutationen der oberen Indices nicht zu ändern, $b_{i_1 i_2 \dots i_r}^{k_1 k_2 \dots k_r}$. Daher lässt sich (13) auch schreiben:

$$Y_{i_1 i_2 \dots i_r} = \sum_{k_1 k_2 \dots k_r}'' [b_{i_1 i_2 \dots i_r}^{k_1 k_2 \dots k_r} (\sum' \Xi_{k_1 k_2 \dots k_r})]. \tag{14}$$

$\sum''_{k_1 k_2 \dots k_r}$ heisst, man soll in der zu bildenden Summe den $k_1, k_2, \cdots, k_r$ nur $\binom{n+r-1}{r}$ Wertecombinationen beilegen, indem man für $k_1, k_2, \cdots, k_r$ immer $r$ der $n$ Zahlen 1, 2, $\cdots$, $n$ ohne Berücksichtigung der Reihenfolge setzt, dabei können die $r$ Zahlen auch mehrfach auftreten. (Combinationen von $n$ Zahlen zu $r$ mit Wiederholung.) Durch (8′) geht (14) über in

$$Y_{i_1 i_2 \dots i_r} = \sum_{k_1 k_2 \dots k_r}'' b_{i_1 i_2 \dots i_r}^{k_1 k_2 \dots k_r} C_{k_1 k_2 \dots k_r} \Upsilon_{k_1 k_2 \dots k_r}. \tag{15}$$

Wir behaupten, (15) ist die $r$te Potenztransformation $P_r A$ der Substitution $A$. Um zur $r$ten Potenztransformation von $A$ zu gelangen, sind in (2) die oberen Indices 1, 2, $\cdots$, $r$ bei $x$ und $\xi$ fortzulassen, es sind *für die rte Potenztransformation* also in (3) $X_{i_1 i_2 \dots i_r}$ und $\Xi_{i_1 i_2 \dots i_r}$ *unabhängig* von der Reihenfolge der Indices anzunehmen. Bei dieser speciellen Annahme wird nach (8) und (8′):

$$Y_{i_1 i_2 \dots i_r} = X_{i_1 i_2 \dots i_r}$$

und

$$\Upsilon_{i_1 i_2 \dots i_r} = \Xi_{i_1 i_2 \dots i_r}.$$

Die Potenztransformation $P_r A$ ergiebt sich daher aus (14), wenn man $Y_{i_1 i_2 \dots i_r} = X_{i_1 i_2 \dots i_r}$ setzt:

$$X_{i_1 i_2 \dots i_r} = \sum_{k_1 k_2 \dots k_r}'' [b_{i_1 i_2 \dots i_r}^{k_1 k_2 \dots k_r} (\sum' \Xi_{k_1 k_2 \dots k_r})]. \tag{16}$$

Für die Potenztransformation ist $\Xi_{k_1 k_2 \dots k_r}$ unabhängig von der Reihenfolge seiner Indices; daher ist

$$\sum_{k_1 k_2 \dots k_r}' \Xi_{k_1 k_2 \dots k_r}$$

für die Potenztransformation eine Summe von $C_{k_1 k_2 \dots k_r}$ gleichen Summanden $\Xi_{k_1 k_2 \dots k_r}$.

Formel (16) geht über in:

$$X_{i_1 i_2 \dots i_r} = \sum_{k_1 k_2 \dots k_r}'' b_{i_1 i_2 \dots i_r}^{k_1 k_2 \dots k_r} \cdot C_{k_1 k_2 \dots k_r} \Xi_{k_1 k_2 \dots k_r}. \tag{17}$$

Formel (17) liefert $P_r A$; bis auf die Variablenbezeichnung ist aber (17) mit der Formel (15) identisch. Hiermit ist der Beweis geleistet.

Wir setzen ferner:

$$T_{l_1 l_2 \dots l_r} = \sum_{l_1 l_2 \dots l_r}''' \epsilon_{l_1 l_2 \dots l_r} X_{l_1 l_2 \dots l_r}. \tag{18}$$

Die Summe

$$\sum_{l_1 l_2 \dots l_r}'''$$

soll alle Summanden umfassen, die man erhält, wenn man die Zahlen $l_1, l_2, \cdots, l_r$ auf jede Art permutirt. Die Zahlen $l_1, l_2, \cdots, l_r$ sollen hierbei $r$ unter einander verschiedene der Zahlen $1, 2, \cdots, n$ sein und zwar sollen die Ausgangszahlen stets in natürlicher Reihenfolge $l_1 < l_2 < l_3 \cdots < l_r$ gewählt werden, so dass bei $T_{l_1 l_2 \dots l_r}$ die Indices in natürlicher Reihenfolge stehen. $T_{l_1 l_2 \dots l_r}$ hat nur eine Bedeutung für $r \leqq n$; den Fall $r = 1$ schliessen wir auch aus, da wir dann die Substitution $A$ selbst haben würden. Verschiedene Grössen $T_{l_1 l_2 \dots l_r}$ giebt es $\binom{n}{r}$ entsprechend der Anzahl der Combinationen von $n$ Zahlen zu $r$ ohne Wiederholungen. Schliesslich bedeutet $\epsilon_{l_1 l_2 \dots l_r}$ die positive Einheit, wenn die Indices $l_1, l_2, \cdots, l_r$ bei dem zugehörigen $X_{l_1 l_2 \dots l_r}$ eine gerade Anzahl von Inversionen aufweisen; weisen die Indices $l_1, l_2, \cdots, l_r$ eine ungerade Anzahl von Inversionen auf, so soll $\epsilon_{l_1 l_2 \dots l_r}$ die negative Einheit sein.

Genau analog zu (18) führen wir ein:

$$T^*_{l_1 l_2 \dots l_r} = \sum_{l_1 l_2 \dots l_r}''' \epsilon_{l_1 l_2 \dots l_r} \Xi_{l_1 l_2 \dots l_r}. \tag{18'}$$

Setzt man in (18) für $X_{l_1 l_2 \dots l_r}$ seinen ursprünglichen Wert $x'^{(1)}_{l_1} \cdot x'^{(2)}_{l_2} \cdots x'^{(r)}_{l_r}$, so wird:

$$T_{l_1 l_2 \dots l_r} = \begin{vmatrix} x'^{(1)}_{l_1} & x'^{(1)}_{l_2} & \cdots & x'^{(1)}_{l_r} \\ x'^{(2)}_{l_1} & x'^{(2)}_{l_2} & \cdots & x'^{(2)}_{l_r} \\ \vdots & \vdots & & \vdots \\ x'^{(r)}_{l_1} & x'^{(r)}_{l_2} & \cdots & x'^{(r)}_{l_r} \end{vmatrix}. \tag{19}$$

Wendet man auf die rechte Seite von (18) die Transformation (3) an, so ist dies gleichbedeutend, als ob man in (19) die $x$ durch die $\xi$ transformirt. Thut man dies, so wird aus (19) erhalten:

$$T_{l_1 l_2 \dots l_r} = \begin{vmatrix} \sum_{s=1}^{s=n} a_{l_1 s} \xi^{(1)}_s & \sum_{s=1}^{s=n} a_{l_2 s} \xi^{(1)}_s & \cdots & \sum_{s=1}^{s=n} a_{l_r s} \xi^{(1)}_s \\ \sum_{s=1}^{s=n} a_{l_1 s} \xi^{(2)}_s & \sum_{s=1}^{s=n} a_{l_2 s} \xi^{(2)}_s & \cdots & \sum_{s=1}^{s=n} a_{l_r s} \xi^{(2)}_s \\ \vdots & \vdots & & \vdots \\ \sum_{s=1}^{s=n} a_{l_1 s} \xi^{(r)}_s & \sum_{s=1}^{s=n} a_{l_2 s} \xi^{(r)}_s & \cdots & \sum_{s=1}^{s=n} a_{l_r s} \xi^{(r)}_s \end{vmatrix}. \tag{20}$$

Nach einem bekannten Determinantensatz kann Formel (20) auch geschrieben werden:

$$(21)\qquad T_{i_1 i_2 \dots i_r} = \sum_{s_1 s_2 \dots s_r}^{(\mathrm{iv})} \begin{vmatrix} a_{i_1 s_1} & a_{i_1 s_2} & \cdots & a_{i_1 s_r} \\ a_{i_2 s_1} & a_{i_2 s_2} & \cdots & a_{i_2 s_r} \\ \vdots & & & \\ a_{i_r s_1} & a_{i_r s_2} & \cdots & a_{i_r s_r} \end{vmatrix} \cdot \begin{vmatrix} \xi^{(1)}_{s_1} & \xi^{(1)}_{s_2} & \cdots & \xi^{(1)}_{s_r} \\ \xi^{(2)}_{s_1} & \xi^{(2)}_{s_2} & \cdots & \xi^{(2)}_{s_r} \\ \vdots & & & \\ \xi^{(r)}_{s_1} & \xi^{(r)}_{s_2} & \cdots & \xi^{(r)}_{s_r} \end{vmatrix}.$$

Die Summe rechter Hand ist über alle Producte zu erstrecken, die erhalten werden, wenn man $s_1, s_2, \cdots, s_r$ auf jede Art unter den Zahlen $1, 2, \cdots, n$ auswählt. $s_1 < s_2 < s_3 \cdots < s_r$. Beachtet man den Wert von $T^*_{i_1 i_2 \dots i_r}$ nach (18′), so wird (21):

$$(22)\qquad T_{i_1 i_2 \dots i_r} = \sum_{s_1 s_2 \dots s_r}^{(\mathrm{iv})} \begin{vmatrix} a_{i_1 s_1} & a_{i_1 s_2} & \cdots & a_{i_1 s_r} \\ a_{i_2 s_1} & a_{i_2 s_2} & \cdots & a_{i_2 s_r} \\ \vdots & & & \\ a_{i_r s_1} & a_{i_r s_2} & \cdots & a_{i_r s_r} \end{vmatrix} \cdot T^*_{s_1 s_2 \dots s_r}.$$

Die Transformation (22) ist aber nichts anderes als die $r$te Determinantentransformation $C_r A$ der Substitution $A$; denn die Grössen $T_{i_1 i_2 \dots i_r}$ sind infolge von (19) (vgl. 4) die Variablen der Substitution $C_r A$.

Die $\binom{n+r-1}{r}$ Variablen $Y_{i_1 i_2 \dots i_r}$ und die $\binom{n}{r}$ Variablen $T_{i_1 i_2 \dots i_r}$, die wir nach (8) und (18) als lineare Combinationen mit constanten Coefficienten aus den $n^r$ Variablen $X_{i_1 i_2 \dots i_r}$ einführten, sind linear unabhängig; denn eine jede lineare Relation zwischen diesen Grössen würde sofort eine solche zwischen den linear unabhängigen $X_{i_1 i_2 \dots i_r}$ zur Folge haben.

Wir ergänzen die Gleichungen (8) und (18) noch durch Einführung von $n^r - \binom{n}{r} - \binom{n+r-1}{r}$ Variablen $U$, die beliebige lineare homogene Functionen der $n^r$ Variablen $X_{i_1 i_2 \dots i_r}$ mit constanten Coefficienten sind und nur so gewählt seien, dass zwischen $Y_{i_1 i_2 \dots i_r}$, $T_{i_1 i_2 \dots i_r}$ und den $U$ keine lineare homogene Relation mit constanten Coefficienten entsteht; hierdurch erhält man eine Substitution $S$ in $n^r$ Variablen. $S$ führt $\Pi_r A$ in die ähnliche Transformation $S \Pi_r A S^{-1}$ über, so dass die ersten $\binom{n+r-1}{r}$ Variablen sich nach der $r$ten Potenztransformation $P_r A$ und die folgenden $\binom{n}{r}$ Variablen nach $C_r A$ transformiren. Führt man $H_r A$ nach (5) ein, so wird:

$$(\text{vgl. } 6)\qquad S \Pi_r A S^{-1} = \begin{matrix} H_r A & 0 \\ A_{21} & A_{22} \end{matrix}.$$

$A_{21}$ und $A_{22}$ bedeuten hierbei Matrices von $n^r - \binom{n+r-1}{r} - \binom{n}{r}$ Zeilen und $\binom{n+r-1}{r} + \binom{n}{r}$ bez. $n^r - \binom{n+r-1}{r} - \binom{n}{r}$ Colonnen.

Für $r > n$ ist $\binom{n}{r} = 0$ und $H_r A = P_r A$.

Die Substitution $S$ hängt nicht von der speciellen Wahl der Substitution $A$ ab; sie ist nur durch $r$ und die Variablenzahl von $A$ bedingt. Durchläuft $A$ die Substitutionen irgend einer Gruppe $G$ linearer homogener Substitutionen,

so existirt also eine feste Matrix $S$ von nicht verschwindender Determinante, dass sämtliche Substitutionen der zu $\Pi_r G$ ähnlichen Gruppe $S\Pi_r G S^{-1}$ von der Form (6) werden.

Die Gruppe $S\Pi_r G S^{-1}$, wobei $S$ eine feste Matrix ist, hat die Form:

$$\begin{matrix} H_r G & 0 \\ G_{21} & G_{22}. \end{matrix}$$

Die *Gruppe* $\Pi_r G$ ist also *reducibel* und weist die *für* $1 < r \leqq n$ *zerlegbare Gruppe* $H_r G$ als einen ihrer Bestandtheile auf. Hiermit ist der in der Einleitung p. 65 angeführte Satz völlig bewiesen.

Für $r = 2$ ist $n' - \binom{n}{r} - \binom{n+r-1}{r} = 0$, daher wird für $r = 2$:

$$S\Pi_2 A S^{-1} = H_2 A = \begin{matrix} P_2 A & 0 \\ 0 & C_2 A \end{matrix}.^*$$

*Die Gruppe* $\Pi_2 G$ *ist in die zerlegbare Gruppe*

$$\begin{matrix} P_2 G & 0 \\ 0 & C_2 G \end{matrix}$$

*transformirbar.*

## § 2.

Die positive ganze Zahl $r$ sei auf irgend welche Art in $r = r_1 + r_2 + \cdots + r_f$, wobei $r_1, r_2, \cdots, r_f$ lauter positive ganze Zahlen bedeuten, zerlegt. Dann ist $\Pi_r A = \Pi_{r_1} A \times \Pi_{r_2} A \times \cdots \times \Pi_{r_f} A$; ist eine der Zahlen $r_e$ $(e = 1, 2, \cdots, f)$ gleich 1, so setzen wir das entsprechende $\Pi_{r_e} A$ einfach gleich $A$.

Bedeutet $e$ irgend eine der Zahlen $1, 2, \cdots, f$, so giebt es nach den Resultaten des § 1 eine Substitution $S_e$ von nicht verschwindender Determinante, dass:

$$S_e \Pi_{r_e} A S_e^{-1} = \begin{pmatrix} H_{r_e} A & 0 \\ A_{21}^{(e)} & A_{22}^{(e)} \end{pmatrix} \qquad (e = 1, 2, \cdots, f) \tag{23}$$

wird; dabei sind $A_{21}^{(e)}$ und $A_{22}^{(e)}$ Matrices von $n^{r_e} - \binom{n+r_e-1}{r_e} - \binom{n}{r_e}$ Zeilen. Nach der mit $(I)$ bezeichneten Eigenschaft der Producttransformation (vgl. S. 61) ist:

$$\begin{aligned} &(S_1 \times S_2 \times \cdots \times S_f)(\Pi_{r_1} A \times \Pi_{r_2} A \times \cdots \times \Pi_{r_f} A) \cdot (S_1^{-1} \times S_2^{-1} \times \cdots \times S_f^{-1}) \\ &\quad = S_1 \Pi_{r_1} A S_1^{-1} \times S_2 \Pi_{r_2} A S_2^{-1} \times \cdots \times S_f \Pi_{r_f} A S_f^{-1}. \end{aligned} \tag{24}$$

Nach der mit $(II)$ bezeichneten Eigenschaft der Producttransformation ist,

$$S_1^{-1} \times S_2^{-1} \times \cdots \times S_f^{-1} = (S_1 \times S_2 \times \cdots \times S_f)^{-1},$$

* Dieses Resultat ist auch aus einer Arbeit von IGEL, *Zur Theorie der Determinanten*, Monatshefte für Mathematik und Physik, Bd. 3 (1892), S. 59, zu entnehmen.

mithin wird (24), wenn man noch für:

$$\Pi_{r_1} A \times \Pi_{r_2} A \times \cdots \times \Pi_{r_f} A = \Pi_r A$$

setzt:

$$(25) \quad \begin{aligned} &(S_1 \times S_2 \times \cdots \times S_f)\,\Pi_r A\,(S_1 \times S_2 \times \cdots \times S_f)^{-1} \\ &\quad = S_1 \Pi_{r_1} A S_1^{-1} \times S_2 \Pi_{r_2} A S_2^{-1} \times \cdots \times S_f \Pi_{r_f} A S_f^{-1}. \end{aligned}$$

Sind $A$ und $B$ irgend welche Matrices der besonderen Form:

$$\begin{matrix} A_{11} & 0 \\ A_{21} & A_{22} \end{matrix} \quad \text{und} \quad \begin{matrix} B_{11} & 0 \\ B_{21} & B_{22} \end{matrix},$$

so sieht man unmittelbar, dass:

$$A \times B = \begin{pmatrix} A_{11} \times B_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

wird: die Matrices $L_{21}$ und $L_{22}$ sind übrigens auch leicht zu bilden. Benützt man diese Bemerkung, so wird vermöge (23) die rechte Seite von (25) gleich der Matrix:

$$(26) \quad \begin{matrix} H_{r_1} A \times H_{r_2} A \times \cdots \times H_{r_f} A & 0 \\ L_{21} & L_{22}. \end{matrix}$$

Die in (25) auftretenden $S_1, S_2, \cdots, S_f$ hängen nur von der Variablenzahl der Substitution $A$ und den positiven Zahlen $r_1, r_2, \cdots, r_f$ ab; sie sind nicht durch die besonderen Werte der Coefficienten des $A$ bedingt, hieraus folgt:

*Durchläuft $A$ irgend eine Gruppe $G$ linearer homogener Substitutionen in $n$ Variablen, und bildet man zu jeder Substitution $A$ von $G$ die zugehörige Substitution $\Pi_r A$ von $\Pi_r G$, so sind alle Substitutionen von $\Pi_r G$ durch eine und dieselbe Substitution in die Form (26) überführbar. Die Gruppe $\Pi_r G$ ist also reducibel und weist $H_{r_1} G \times H_{r_2} G \times \cdots \times H_{r_f} G$ als Bestandteil oder Teilgruppe auf; dabei ist der Substitution $\Pi_r A$ von $\Pi_r G$ die Substitution $H_{r_1} A \times H_{r_2} A \times \cdots \times H_{r_f} A$ von $H_{r_1} G \times H_{r_2} G \times \cdots \times H_{r_f} G$ zugeordnet.*

Dieser Satz enthält den des § 1 als Specialfall, wie sich ergiebt, wenn man $r_1 = r$ wählt.

Bildet man das Product der zwei zerfallenden Matrices:

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix},$$

so wird dieses:

$$(27) \quad \begin{Vmatrix} A_{11} \times B_{11} & 0 & 0 & 0 \\ 0 & A_{11} \times B_{22} & 0 & 0 \\ 0 & 0 & A_{22} \times B_{11} & 0 \\ 0 & 0 & 0 & A_{22} \times B_{22} \end{Vmatrix};$$

ferner ist das Product einer zerfallenden Matrix in eine nicht zerfallende:

$$(28) \qquad \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \times B = \begin{pmatrix} A_{11} \times B & 0 \\ 0 & A_{22} \times B \end{pmatrix}.$$

Schliesst man $r_1 = r_2 = \cdots r_f = 1$ oder sämtliche Zahlen $r_1, r_2, \cdots, r_f$ gleichzeitig grösser als $n$ aus, so ist $H_{r_1}G \times H_{r_2}G \times \cdots \times H_{r_f}G$ nach (5) ein Product von Gruppen, von denen wenigstens eine zerfällt. Beachtet man (27) und (28), so erhält man das Resultat:

*$H_{r_1}G \times H_{r_2}G \times \cdots \times H_{r_f}G$ ist, wenn die zwei Fälle $r_1 = r_2 = \cdots = r_f = 1$ und $r_1 > n$ gleichzeitig $r_2 > n, r_3 > n, \cdots, r_f > n$ ausgeschlossen werden, eine zerfallende Gruppe. In der Diagonale treten bei $H_{r_1}G \times H_{r_2}G \times \cdots \times H_{r_f}G$ nur die Gruppen $K_{r_1}G \times K_{r_2}G \times \cdots \times K_{r_f}G$ auf, wobei für $K$ sowohl $P$ wie $C$ zu wählen sind. $K_1G$ ist einfach $G$, bei $K_{r_\varepsilon}G$ ist für $r_\varepsilon \geqq n+1$ nur $P_{r_\varepsilon}G$ zu setzen.*

Mit Hülfe dieses Satzes beweisen wir, dass $\Pi_r G$ *auch stets einer zerfallenden Gruppe* ähnlich ist. Wählt man nämlich $r_1 = r_2 = \cdots = r_{f-1} = 2$ und $r_f = 1$ oder 2, je nachdem $r$ eine ungerade oder gerade Zahl ist, so besitzen für diese besondere Wahl der Zahlen $r_1, r_2, \cdots, r_f$ die Substitutionen $\Pi_r A$ und $H_{r_1}A \times H_{r_2}A \times \cdots \times H_{r_f}A$ gleichviel Variablen. $H_2A$ ist nämlich eine Matrix von

$$\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$$

Variablen, $H_{r_f}A$ ist für $r_f = 1$ mit $A$ identisch. Für die angegebene Wahl der Zahlen $r_1, r_2, \cdots, r_f$ wird die rechte Seite von (26) einfach $H_{r_1}A \times H_{r_2}A \times \cdots \times H_{r_f}A$, ferner zerfällt $H_{r_1}G \times H_{r_2}G \times \cdots \times H_{r_f}G$ nach dem zuletzt bewiesenen Satze. Wir haben daher das Resultat:

*Ist $r$ eine gerade Zahl, so ist $\Pi_r G$ zu der zerfallenden Gruppe $H_2G \times H_2G \times \cdots \times H_2G$ ähnlich; ist $r$ eine ungerade Zahl, so ist $\Pi_r G$ zu der zerfallenden Gruppe $H_2G \times H_2G \times \cdots \times H_2G \times G$ ähnlich. Im ersten Falle entspricht der Substitution $\Pi_r A$ die Substitution $H_2A \times H_2A \times \cdots \times H_2A$, im zweiten Falle die Substitution $H_2A \times H_2A \times \cdots \times H_2A \times A$.*

## § 3.

Es sei eine lineare homogene Differentialgleichung:

$$(D) \qquad \frac{d^n x}{dt^n} + p_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_n(t)x = 0$$

gegeben; der Einfachheit wegen nehmen wir an, dass $p_1(t), p_2(t), \cdots, p_n(t)$ rationale Functionen von $t$ sind.

Wir verwenden die folgenden Hülfssätze:

I. Seien $x_1, x_2, \cdots, x_n$ ein Fundamentalsystem von Integralen von $(D)$ und $\eta_1, \eta_2, \cdots, \eta_m$ rationale Differentialfunctionen von $x_1, x_2, \cdots, x_n$ (d. h. rationale Functionen von $x_1, x_2, \cdots, x_n$ und deren Abgeleiteten mit rationalen Coefficienten), die bei jeder linearen homogenen Transformation der $x_1, x_2, \cdots, x_n$ sich ebenfalls linear homogen mit constanten Coefficienten transformiren, so sind, falls zwischen den $m$ Functionen $\eta_1, \eta_2, \cdots, \eta_m$, welche Functionen von $t$ sind, keine lineare homogene Relation mit constanten, also von $t$ unabhängigen Coefficienten besteht, die Functionen $\eta_1, \eta_2, \cdots, \eta_m$ die Elemente eines Fundamentalsystemes einer linearen homogenen Differentialgleichung $m$ter Ordnung mit rationalen Coefficienten; bestehen zwischen $\eta_1, \eta_2, \cdots, \eta_m$ $\mu \geqq 1$ lineare homogene, linear unabhängige Relationen mit constanten, also von $t$ unabhängigen Coefficienten, so bilden $m - \mu$ der Functionen $\eta_1, \eta_2, \cdots, \eta_m$ ein Fundamentalsystem von Integralen einer linearen homogenen Differentialgleichung $m - \mu$ter Ordnung mit rationalen Coefficienten.

Zwischen $\eta_1, \eta_2, \cdots, \eta_m$ mögen $\mu \geqq 1$ linear unabhängige, lineare homogene Relationen mit von $t$ unabhängigen, also constanten Coefficienten bestehen. Sie mögen lauten:

$$\begin{aligned} v_1 &= g_{11}\eta_1 + g_{12}\eta_2 + \cdots + g_{1m}\eta_m = 0, \\ v_2 &= g_{21}\eta_1 + g_{22}\eta_2 + \cdots + g_{2m}\eta_m = 0, \\ &\vdots \\ v_\mu &= g_{\mu 1}\eta_1 + g_{\mu 2}\eta_2 + \cdots + g_{\mu m}\eta_m = 0. \end{aligned}$$

Damit die erste Relation eine wirkliche Relation ist, wird in ihr sicher wenigstens eines der $\eta$ einen von 0 verschiedenen Coefficienten haben. Wir können uns daher, die Functionen $\eta_1, \eta_2, \cdots, \eta_m$ so bezeichnet denken, dass $g_{1m} \neq 0$ wird. Wir bilden dann:

$$v_2' = g_{2m}v_1 - g_{1m}v_2, \quad v_3' = g_{3m}v_1 - g_{1m}v_3, \cdots, v_\mu' = g_{\mu m}v_1 - g_{1m}v_\mu.$$

Keine der Functionen $v_2', v_3', \cdots, v_\mu'$ enthält $\eta_m$, daher werden diese Functionen von der Form:

$$\begin{aligned} v_2' &= g_{21}'\eta_1 + g_{22}'\eta_2 + \cdots + g_{2m-1}'\eta_{m-1}, \\ v_3' &= g_{31}'\eta_1 + g_{32}'\eta_2 + \cdots + g_{3m-1}'\eta_{m-1}, \\ &\vdots \\ v_\mu' &= g_{\mu 1}'\eta_1 + g_{\mu 2}'\eta_2 + \cdots + g_{\mu m-1}'\eta_{m-1}, \end{aligned}$$

wobei die $g'$ Constante sind.

Wegen der linearen Unabhängigkeit von $v_1, v_2, \cdots, v_\mu$ können $v_2', v_3', \cdots, v_\mu'$ nicht identisch verschwinden; jede dieser Functionen enthält daher wenigstens ein $\eta$ mit von Null verschiedenem Coefficienten. Wir denken uns daher die gegebenen Functionen $\eta$ so bezeichnet, dass der Coefficient $g_{2m-1}'$ von $\eta_{m-1}$ in $v_2'$

von Null verschieden ist. $v_2', v_3', \cdots, v_\mu'$ sind linear unabhängig; denn angenommen, es gäbe Constante $\sigma_2, \sigma_3, \cdots, \sigma_\mu$, welche nicht sämtlich Null sind, dass: $\sigma_2 v_2' + \sigma_3 v_3' + \cdots + \sigma_\mu v_\mu' = 0$ wird, so würde eine Relation:

$$(\sigma_2 g_{2m} + \sigma_3 g_{3m} + \cdots + \sigma_\mu g_{\mu m}) v_1 - \sigma_2 g_{1m} v_2 - \sigma_3 g_{1m} v_3 \cdots - \sigma_\mu g_{1m} v_\mu = 0$$

folgen. Da $g_{1m} \neq 0$ ist, so würden auch $v_1, v_2, \cdots, v_\mu$ in linearer Dependenz stehen, also $v_1 = 0, v_2 = 0, \cdots, v_\mu = 0$ im Gegensatz zur Voraussetzung nicht $\mu$ linear unabhängige Relationen darstellen. Die $\mu$ Functionen $v_1, v_2', v_3', \cdots v_\mu'$ sind auch linear unabhängig; denn $v_1$ enthält $\eta_m$, das in $v_2', v_3', \cdots, v_\mu'$ nicht auftritt, ferner sind $v_2', v_3', \cdots, v_\mu'$ linear unabhängig. Leitet man aus $v_2', v_3', \cdots, v_\mu'$:

$$v_3'' = g_{3m-1}' v_2' - g_{2m-1}' v_3', \quad v_4'' = g_{4m-1}' v_2' - g_{2m-1}' v_4', \cdots,$$

$$v_\mu'' = g_{\mu m-1}' v_2' - g_{2m-1}' v_\mu' ab,$$

so erhält man von $\eta_{m-1}$ freie Functionen.

Fährt man auf diese Art fort, so ergiebt sich: Man kann die $m$ Functionen $\eta_1, \eta_2, \cdots, \eta_m$, zwischen denen $\mu$ linear unabhängige, lineare homogene Relationen mit constanten Coefficienten bestehen, so angeordnet denken, dass diese Relationen lauten:

$$\begin{aligned}
v_1 &= g_{11}\eta_1 + g_{12}\eta_2 + \cdots + g_{1m}\eta_m = 0, \\
v_2' &= g_{21}'\eta_1 + g_{22}'\eta_2 + \cdots + g_{2m-1}'\eta_{m-1} = 0, \\
v_3'' &= g_{31}''\eta_1 + g_{32}''\eta_2 + \cdots + g_{3m-2}''\eta_{m-2} = 0, \\
&\vdots \\
v_\mu^{(\mu-1)} &= g_{\mu 1}^{(\mu-1)}\eta_1 + g_{\mu 2}^{(\mu-1)}\eta_2 + \cdots + g_{\mu\, m-\mu+1}^{(\mu-1)}\eta_{m-\mu+1} = 0.
\end{aligned}$$

Dabei sind die $\mu$ Constanten $g_{1m}, g_{2m-1}', g_{3m-2}'', \cdots, g_{\mu\, m-\mu+1}^{(\mu-1)}$ von Null verschieden. Aus diesen Gleichungen kann man daher $\eta_{m-\mu+1}, \eta_{m-\mu+2}, \cdots, \eta_m$ in der Form

$$\begin{aligned}
\eta_{m-\mu+1} &= c_{11}\eta_1 + c_{12}\eta_2 + \cdots + c_{1m-\mu}\eta_{m-\mu}, \\
\eta_{m-\mu+2} &= c_{21}\eta_1 + c_{22}\eta_2 + \cdots + c_{2m-\mu}\eta_{m-\mu}, \\
&\vdots \\
\eta_m &= c_{\mu 1}\eta_1 + c_{\mu 2}\eta_2 + \cdots + c_{\mu m-\mu}\eta_{m-\mu},
\end{aligned} \qquad (c)$$

finden, wobei die $c$ Constante bedeuten; da nur $\mu$ Relationen bestehen sollen, so sind $\eta_1, \eta_2, \cdots, \eta_{m-\mu}$ linear unabhängig. Die gegebenen $m$ rationalen Differentialfunctionen $\eta_1, \eta_2, \cdots, \eta_m$, zwischen denen $\mu$ linear unabhängige Relationen bestehen, seien so angeordnet, dass $\eta_1, \eta_2, \cdots, \eta_{m-\mu}$ linear unabhängig sind und die $\mu$ Relationen in Form der Gleichungen $(c)$ gegeben seien. Wegen der linearen Unabhängigkeit von $\eta_1, \eta_2, \cdots, \eta_{m-\mu}$ verschwindet die Wron-

skische Determinante dieser Grössen, die wir mit $\Delta$ bezeichnen, nicht. Wir bilden dann:

$$\begin{vmatrix} \eta & \frac{d\eta}{dt} & \frac{d^2\eta}{dt^2} & \cdots & \frac{d^{m-\mu}\eta}{dt^{m-\mu}} \\ \eta_1 & \frac{d\eta_1}{dt} & \frac{d^2\eta_1}{dt^2} & \cdots & \frac{d^{m-\mu}\eta_1}{dt^{m-\mu}} \\ \eta_2 & \frac{d\eta_2}{dt} & \frac{d^2\eta_2}{dt^2} & \cdots & \frac{d^{m-\mu}\eta_2}{dt^{m-\mu}} \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_{m-\mu} & \frac{d\eta_{m-\mu}}{dt} & \frac{d^2\eta_{m-\mu}}{dt^2} & \cdots & \frac{d^{m-\mu}\eta_{m-\mu}}{dt^{m-\mu}} \end{vmatrix} : \Delta = 0 .$$

Entwickelt man linker Hand nach den Elementen der ersten Zeile der im Zähler stehenden Determinante, so wird man auf die lineare homogene Differentialgleichung:

$$\frac{d^{m-\mu}\eta}{dt^{m-\mu}} - q_1 \frac{d^{m-\mu-1}\eta}{dt^{m-\mu-1}} + q_2 \frac{d^{m-\mu-2}\eta}{dt^{m-\mu-2}} - \cdots + (-1)^{m-\mu-1} q_{m-\mu-1} \frac{d\eta}{dt} + (-1)^{m-\mu} q_{m-\mu}\eta = 0 ,$$

$m-\mu$ter Ordnung in $\eta$ geführt; dabei sind $q_1, q_2, \cdots, q_{m-\mu}$ Determinantenquotienten. Betrachten wir z. B.

$$q_{m-\mu} = \begin{vmatrix} \frac{d\eta_1}{dt} & \frac{d^2\eta_1}{dt^2} & \cdots & \frac{d^{m-\mu}\eta_1}{dt^{m-\mu}} \\ \frac{d\eta_2}{dt} & \frac{d^2\eta_2}{dt^2} & \cdots & \frac{d^{m-\mu}\eta_2}{dt^{m-\mu}} \\ \vdots & \vdots & & \vdots \\ \frac{d\eta_{m-\mu}}{dt} & \frac{d^2\eta_{m-\mu}}{dt^2} & \cdots & \frac{d^{m-\mu}\eta_{m-\mu}}{dt^{m-\mu}} \end{vmatrix} : \begin{vmatrix} \eta_1 & \frac{d\eta_1}{dt} & \cdots & \frac{d^{m-\mu-1}\eta_1}{dt^{m-\mu-1}} \\ \eta_2 & \frac{d\eta_2}{dt} & \cdots & \frac{d^{m-\mu-1}\eta_2}{dt^{m-\mu-1}} \\ \vdots & \vdots & & \vdots \\ \eta_{m-\mu} & \frac{d\eta_{m-\mu}}{dt} & \cdots & \frac{d^{m-\mu-1}\eta_{m-\mu}}{dt^{m-\mu-1}} \end{vmatrix} .$$

Transformirt man $x_1, x_2, \cdots, x_n$ linear homogen mit constanten Coefficienten, so sollen sich nach Voraussetzung auch $\eta_1, \eta_2, \cdots, \eta_m$ linear homogen mit constanten Coefficienten transformiren. Möge bei einer linearen homogenen Transformation der $x_1, x_2, \cdots, x_n$ sich $\eta_i$ in

$$b_{i1}\eta_1 + b_{i2}\eta_2 + \cdots + b_{im}\eta_m \qquad (i = 1, 2, \cdots, m)$$

transformiren, dann transformiren sich die Abgeleiteten von $\eta_1, \eta_2, \cdots, \eta_m$ cogredient mit $\eta_1, \eta_2, \cdots, \eta_m$. Darf man die Relationen $(c)$ benützen, so verhalten sich $\eta_1, \eta_2, \cdots, \eta_{m-\mu}$ auf folgende Art; es geht $\eta_i$ über in

$$b'_{i1}\eta_1 + b'_{i2}\eta_2 + \cdots + b'_{im-\mu}\eta_{m-\mu} \qquad (i = 1, 2, \cdots, m-\mu).$$

Hierbei ist:

$$b'_{ik} = b_{ik} + b_{im}c_{\mu k} + b_{im-1}c_{\mu-1k} + b_{im-2}c_{\mu-2k} + \cdots + b_{im-\mu+1}c_{1k}$$
$$(i = 1, 2, \cdots m-\mu\,;\; k = 1, 2, \cdots, m-\mu).$$

Da für die Abgeleiteten von $\eta_1, \eta_2, \cdots, \eta_m$ auch die Relationen $(c)$ bestehen, so transformiren sich die Abgeleiteten von $\eta_1, \eta_2, \cdots, \eta_{m-\mu}$ auch noch in diesem Falle cogredient zu $\eta_1, \eta_2, \cdots, \eta_{m-\mu}$.

Bei einer jeden linearen homogenen Transformation der $x_1, x_2, \cdots, x_n$, welche eine solche der $\eta_1, \eta_2, \cdots, \eta_m$ nach sich zieht und bei der auch die Relationen $(c)$ verwandt werden dürfen, multiplicirt sich Zähler und Nenner rechter Hand bei $q_{m-\mu}$ mit demselben Factor, nämlich der Determinante $|b'_{ik}|\binom{i=1,2\ldots,m-\mu}{k=1,2\ldots,m-\mu}$; die rechte Seite von $q_{m-\mu}$ ändert sich also nicht. Untersucht man das Verhalten irgend einer rationalen Differentialfunction des Fundamentalsystemes $x_1, x_2, \cdots, x_n$ bezüglich der Transformationen der Rationalitätsgruppe von $(D)$, so hat man nur zu achten, wie diese rationale Differentialfunction sich als Function der unabhängigen Variablen $t$ bei den Transformationen der Rationalitätsgruppe verhält. Es handelt sich also bei Anwendung der Picard-Vessiotschen Theorie nie um formale Invarianz. Untersucht man daher die rechte Seite von $q_{m-\mu}$, welche auch eine rationale Differentialfunction von $x_1, x_2, \cdots, x_n$ ist, weil die $\eta$ nach Annahme rationale Differentialfunctionen von $x_1, x_2, \cdots, x_n$ sein sollen, so können für die Betrachtung der Änderungen, welche die rechte Seite von $q_{m-\mu}$ durch die Transformationen der Rationalitätsgruppe von $(D)$ erfährt, die Relationen $(c)$ verwandt werden. Benützt man aber die Relationen $(c)$, so ändert sich nach dem Voraufgehenden die rechte Seite von $q_{m-\mu}$ bei keiner linearen homogenen Transformation der $x$, also auch nicht bei denjenigen der Rationalitätsgruppe von $(D)$ und bleibt daher bei diesen Transformationen dieselbe Function von $t$; folglich ist nach dem bekannten Picard-Vessiotschen Fundamentalsatz * $q_{m-\mu}$ eine rationale Function von $t$. Ebenso zeigt man, dass $q_1, q_2, \cdots, q_{m-\mu-1}$ rationale Functionen von $t$ sind.

Sollte zwischen $\eta_1, \eta_2, \cdots, \eta_m$ keine lineare homogene Relation mit constanten Coefficienten stattfinden, so folgt der zu beweisende Satz schon ohne Verwendung der Rationalitätsgruppe aus dem bekannten Satze des Herrn APPELL †

* Vgl. PICARD, *Traité d'analyse*, t. 3, S. 537, Paris, 1896.

† Vgl. APPELL, *Mémoire sur les équations différentielles linéaires*, Annales de l'école normale supérieure, ser. 2, t. 10 (1881), p. 408; siehe auch E. BEKE, *Die symmetrischen Functionen bei den linearen homogenen Differentialgleichungen*, Math. Annalen, Bd. 45 (1894), S. 295. Die Untersuchungen von Herrn APPELL setzen lineare Unabhängigkeit von $\eta_1, \eta_2, \cdots, \eta_m$ voraus. Bei linearer Abhängigkeit der $\eta_1, \eta_2, \cdots, \eta_m$ verschwindet nicht nur die Wronskische Determinante der genannten Functionen, sondern auch jede Determinante:

$$\begin{vmatrix} \eta_1^{(\lambda_1)} & \eta_2^{(\lambda_1)} & \cdots & \eta_m^{(\lambda_1)} \\ \eta_1^{(\lambda_2)} & \eta_2^{(\lambda_2)} & \cdots & \eta_m^{(\lambda_2)} \\ \vdots & & & \\ \eta_1^{(\lambda_m)} & \eta_2^{(\lambda_m)} & \cdots & \eta_m^{(\lambda_m)} \end{vmatrix},$$

wobei $\eta_i^{(\lambda_k)} = d^{\lambda_k}\eta_i/dt^{\lambda_k}$ ist, und $\lambda_1, \lambda_2, \cdots, \lambda_m$ irgend $m$ positive ganze Zahlen bedeuten.

über die invarianten Functionen einer linearen homogenen Differentialgleichung.

(II). Sind $\zeta_1, \zeta_2, \cdots, \zeta_m$ $m$ weitere rationale Differentialfunctionen von $x_1, x_2, \cdots, x_n$, die sich bei sämtlichen linearen homogenen Transformationen von $x_1, x_2, \cdots, x_n$ cogredient zu $\eta_1, \eta_2, \cdots, \eta_m$ transformiren, und sind $\zeta_1, \zeta_2, \cdots, \zeta_m$ nicht in linearer Dependenz, so besteht ein System von Gleichungen:

$$\eta_i = d_0(t)\zeta_i + d_1(t)\frac{d\zeta_i}{dt} + \cdots + d_{m-1}(t)\frac{d^{m-1}\zeta_i}{dt^{m-1}} \qquad (i = 1, 2, \cdots, m),$$

dabei bedeuten $d_0(t), d_1(t), \cdots, d_{m-1}(t)$ rationale Functionen von $t$.

Die lineare homogene Differentialgleichung mit rationalen Coefficienten, welche die Functionen $\eta$ zu Integralen hat, ist also mit der linearen homogenen Differentialgleichung mit rationalen Coefficienten, welche die Functionen $\zeta$ zu Integralen hat, *von derselben Art.* *

Der eben angegebene Satz (II) ist eine unmittelbare Folge des schon erwähnten Theoremes von Herrn APPELL; denn $d_k(t)$ $(k = 0, 1, 2, \cdots, m-1)$ ist als Quotient $\Delta_k/\Delta$ zweier Determinanten darstellbar; dieser Quotient bleibt bei allen linearen homogenen Transformationen der $x_1, x_2, \cdots, x_n$ formal ungeändert, da sich $\eta_1, \eta_2, \cdots, \eta_m$ und $\zeta_1, \zeta_2, \cdots, \zeta_m$ stets cogredient transformiren und $\Delta_k$ wie $\Delta$ sich daher mit demselben Factor multipliciren. $\Delta$ ist als Wronskische Determinante der linear unabhängigen Functionen $\zeta_1, \zeta_2, \cdots, \zeta_m$ von Null verschieden.

Die Anwendung der zwei Hülfssätze liefert die folgenden Theoreme:

*Bildet man sich die $n^r$ Functionen:*

$$X^{\lambda_1\lambda_2\cdots\lambda_r}_{i_1 i_2 \cdots i_r} = \frac{d^{\lambda_1}x_{i_1}}{dt^{\lambda_1}} \cdot \frac{d^{\lambda_2}x_{i_2}}{dt^{\lambda_2}} \cdots \frac{d^{\lambda_r}x_{i_r}}{dt^{\lambda_r}},$$

*wobei $\lambda_1, \lambda_2, \cdots, \lambda_r$ irgend $r$ feste, verschiedene ganze positive Zahlen sind, $i_1, i_2, \cdots, i_r$ jede der Zahlen $1, 2, \cdots, n$ bedeuten, $x_1, x_2, \cdots, x_n$ ein Fundamentalsystem von Integralen einer linearen homogenen Differentialgleichung $(D)$ nter Ordnung mit rationalen Coefficienten sind, so stellen die linear unabhängigen unter den $n^r$ Functionen $X^{\lambda_1\lambda_2\cdots\lambda_r}_{i_1 i_2 \cdots i_r}$ ein Fundamentalsystem von Integralen einer linearen homogenen Differentialgleichung der Ordnung $\leqq n^r$ mit ebenfalls rationalen Coefficienten dar. Diese Differentialgleichung sei mit $\Pi_r^{\lambda_1\lambda_2\cdots\lambda_r}(D) = 0$ bezeichnet. Betrachtet man die unendliche Anzahl dieser linearen homogenen Differentialgleichungen $\Pi_r^{\lambda_1\lambda_2\cdots\lambda_r}(D) = 0$, welche zu der linearen homogenen Differentialgleichung $D = 0$ gehören und den verschiedenen möglichen Werten für die unter einander verschiedenen ganzen positiven Zahlen $\lambda_1, \lambda_2, \cdots, \lambda_r$ entsprechen, so ist jede dieser Differential-*

* Vgl. meine Arbeit, *Über reducible lineare homogene Differentialgleichungen* in den Mathematischen Annalen, Bd. 56 (1903), S. 554.

*gleichungen* $\Pi_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$ *mit einer jeden unter denselben, deren Ordnung wirklich die Maximalzahl* $n^r$ *erreicht, von derselben Art.*

*Bildet man sich in Analogie mit* (8) *aus den* $n^r$ *Functionen* $X_{i_1i_2\cdots i_r}^{\lambda_1\lambda_2\cdots\lambda_r}$ *die* $\binom{n+r-1}{r}$ *Functionen:*

$$Y_{i_1i_2\cdots i_r}^{\lambda_1\lambda_2\cdots\lambda_r}=\frac{\sum' X_{i_1i_2\cdots i_r}^{\lambda_1\lambda_2\cdots\lambda_r}}{C_{i_1i_2\cdots i_r}},$$

*so stellen die linear unabhängigen derselben ein Fundamentalsystem von Integralen einer linearen homogenen Differentialgleichung mit rationalen Coefficienten der Ordnung* $\leqq\binom{n+r-1}{r}$ *dar. Diese Differentialgleichung sei mit* $P_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$ *bezeichnet.*

*Jede der unendlich vielen linearen homogenen Differentialgleichungen* $P_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$, *die man zu der Differentialgleichung* $D=0$ *bilden kann, ist mit jeder Differentialgleichung* $P_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$, *deren Ordnung wirklich die Maximalzahl* $\binom{n+r-1}{r}$ *besitzt, von derselben Art.*

*Bildet man sich in Analogie zu* (18) *aus den* $n^r$ *Functionen* $X_{i_1i_2\cdots i_r}^{\lambda_1\lambda_2\cdots\lambda_r}$ *die* $\binom{n}{r}$ *Functionen:*

$$T_r^{\lambda_1\lambda_2\cdots\lambda_r}=\sum_{i_1i_2\cdots i_r}''' \epsilon_{i_1i_2\cdots i_r}X_{i_1i_2\cdots i_r}^{\lambda_1\lambda_2\cdots\lambda_r},$$

*wobei* $r\leqq n$ *ist, so stellen die linear unabhängigen unter den* $\binom{n}{r}$ *Functionen ein Fundamentalsystem von Integralen einer linearen homogenen Differentialgleichung mit rationalen Coefficienten der Ordnung* $\leqq\binom{n}{r}$ *dar. Diese Differentialgleichung sei mit* $C_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$ *bezeichnet.*

Die Differentialgleichung $C_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$ ist für den speciellen Fall $\lambda_1=0, \lambda_2=1, \cdots, \lambda_r=r-1$ in der Litteratur wohl bekannt. Die *Differentialgleichung* $C_r^{0,1,2,\ldots,r-1}(D)=0$ wird von Herrn LUDW. SCHLESINGER im Anschluss an Herrn FORSYTH die $n-r$*te associierte Differentialgleichung von* $D=0$ *genannt.*

*Jede der unendlich vielen linearen homogenen Differentialgleichungen* $C_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$, *die man zu der Differentialgleichung* $D=0$ *bilden kann, ist mit jeder Differentialgleichung* $C_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$, *deren Ordnung die Maximalzahl* $\binom{n}{r}$ *besitzt, von derselben Art.* *

*Die lineare homogene Differentialgleichung* $\Pi_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$ *ist stets reducibel; sie wird durch sämtliche Integrale sowohl von* $P_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$ *als auch von* $C_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)=0$, *wenn diese letztere Differentialgleichung existirt, befriedigt.*

Ist $\lambda$ irgend eine ganze positive Zahl, so sind die linear unabhängigen unter den $\binom{n+r-1}{r}$ Functionen:

$$\frac{d^\lambda x_{i_1}}{dt^\lambda}\cdot\frac{d^\lambda x_{i_2}}{dt^\lambda}\cdots\frac{d^\lambda x_{i_r}}{dt^\lambda},$$

* Den obigen Satz in dem speciellen Falle, dass $\lambda_1, \lambda_2, \ldots, \lambda_r$ aus den Zahlen 0, 1, 2, $\ldots$, $n-1$ entnommen werden, behandelt Herr LUDW. SCHLESINGER in seinem *Handbuch der Theorie der linearen Differentialgleichungen*, $\mathrm{II}_1$, S. 127, auf andere Art.

die man erhält, wenn $i_1, i_2, \cdots, i_r$ jede der Zahlen $1, 2, \cdots, n$ bedeuten, auch ein Fundamentalsystem von Integralen einer linearen homogenen Differentialgleichung mit rationalen Coefficienten der Ordnung $\leqq \binom{n+r-1}{r}$.

*Diese Differentialgleichung sei mit* $P_r^{\lambda\lambda\cdots\lambda}(D) = 0$ *bezeichnet.*

*Jede der linearen homogenen Differentialgleichungen* $P_r^{\lambda_1\lambda_2\cdots\lambda_r}(D)$, *wobei* $\lambda_1, \lambda_2, \cdots, \lambda_r$ $r$ *verschiedene ganze positive Zahlen sind, und jede der linearen homogenen Differentialgleichungen* $P_r^{\lambda\lambda\cdots\lambda}(D) = 0$ *ist sowohl mit jeder der Differentialgleichungen* $P_r^{\lambda_1\lambda_2\cdots\lambda_r}(D) = 0$ *als auch mit jeder der Differentialgleichungen* $P_r^{\lambda\lambda\cdots\lambda}(D) = 0$, *welche die Maximalordnung* $\binom{n+r-1}{r}$ *thatsächlich erreichen, von derselben Art.*

Besonders die Differentialgleichung $P_r^{000\cdots0}(D) = 0$, welche $\lambda = 0$ entspricht, und $x_1^r, x_2^r, \cdots, x_{i_1} \cdot x_{i_2} \cdots x_{i_r}, \cdots, x_n^r$ zu Integralen hat, ist vielfach in der Litteratur * behandelt. †

FREIBURG I. B., *Juni* 1903.

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* Vgl. LUDW. SCHLESINGER's, Handbuch, $II_1$, S. 201; doch ist dort bloss der Nachweis geführt, dass, wenn zwischen $x_1, x_2, \ldots, x_n$ keine homogene ganze algebraische Beziehung $r$ten Grades mit constanten Coefficienten besteht, $P_r^{00\cdots0}(D) = 0$ eine lineare homogene Differentialgleichung der Ordnung $\binom{n+r-1}{r}$ ist. Ich darf noch bemerken, dass für den einfachsten Fall $r = 2$ ich die obigen Resultate des § 3 bereits in einer Comptes Rendus Note der Pariser Academie vom 30. December 1901 zur Sprache gebracht habe.

† NACHBEMERKUNG (*November* 1903): Das in dem Bulletin of the American Mathematical Society, vol. 10 (1903), S. 65 erstattete Referat über die vorliegende Arbeit enthält die Worte: "Professor LOEWY shows that the group $\Pi_r G$ is always reducible, $P_r G$ and $C_r G$ being two of its irreducible constituents." Hierdurch werde ich zu folgender Bemerkung veranlasst: Ich habe in der vorstehend erschienenen Arbeit nur gezeigt, dass $P_r G$ und $C_r G$ *Bestandteile oder Teilgruppen* von $\Pi_r G$, *nicht aber irreducible Bestandteile oder irreducible Teilgruppen* von $\Pi_r G$ im Sinne meiner im vierten Bande dieser Transactions erschienenen Arbeit, S. 46 sind. Es ist also im fraglichen Referate das Wort "irreducible" zu streichen. Mehr aber, als dass $P_r G$ und $C_r G$ Bestandteile von $\Pi_r G$ sind, kann man auch nicht beweisen. Es gilt vielmehr folgender Satz, dessen Beweis unschwer erbracht werden kann: *Ist $G$ eine reducible Gruppe, so sind alle Gruppen $P_r G$ und alle Gruppen $C_r G$ die letzteren für $r < n$, wenn $G$ eine Gruppe in $n$ Variablen ist, reducibel.*

# ON THE GROUP OF SIGN $(0, 3; 2, 4, \infty)$ AND THE FUNCTIONS BELONGING TO IT*

BY

JOHN WESLEY YOUNG

## *Introduction.*

The group $(\Gamma)$ which is the object of the present investigation has already been briefly discussed by HURWITZ.† His paper deals chiefly with the groups of signs $(0, 3; 2, 4, \infty)$ and $(0, 3; 2, 6, \infty)$.‡ He proves them "commensurable" with the modular-group $(0, 3; 2, 3, \infty)$ and hence derives certain algebraic relations between the simplest functions belonging to his groups on the one hand and the invariant $J$ of the modular group on the other. He also obtains the arithmetic character of the substitutions of his groups. The group $\Gamma$ is also mentioned briefly in FRICKE-KLEIN'S treatise,§ where it appears as the "reproducing group" of a certain ternary quadratic form.

In Part I, we derive $\Gamma$ as the monodromy group of the Riemann surface $R$ $(p = 2)$ defined by the equation

$$y^4 = (x - \kappa_1)^2(x - \kappa_2)^2(x - \kappa_3)(x - \kappa_4)^3,$$

i. e., as the group on the parameter $z$ involved in the periods of two independent normal integrals of the first kind on $R$, generated by the movement of the branch-points about any closed paths. ‖

Part II contains a rather detailed discussion of $\Gamma$ by the methods applied to the

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† *Über eine Reihe neuer Functionen, etc.*, Mathematische Annalen, vol. 20 (1882), pp. 125–134.

‡ The latter has been discussed by HUTCHINSON (cf. reference below). For notation cf. FRICKE-KLEIN, *Automorphe Functionen*, vol. 1, p. 383.

§ *Automorphe Functionen*, vol. 1, p. 548.

‖ This and another surface $[y^3 = (x - k_1)(x - k_2)(x - k_3)^2(x - k_4)^2]$ have already received notice by BURKHARDT, *Über die Darstellung einiger Fälle der automorphen Primformen durch specielle Thetareihen*, Mathematische Annalen, vol. 42 (1893), pp. 184–214. The monodromy group of the latter surface and the corresponding automorphic functions have been discussed by HUTCHINSON, *On a class of automorphic functions*, Transactions of the American Mathematical Society, vol. 3 (1902), pp. 1–11.

modular group in FRICKE-KLEIN's treatise.* In view of the commensurability of the two groups and the similarity of their generating substitutions, we are not surprised to find many properties common to $\Gamma$ and the modular group. The most noteworthy difference is perhaps the fact that, whereas the latter contains only a finite number (six) of invariant subgroups of genus zero, $\Gamma$ contains a simply infinite system of such subgroups of index $2n$; the corresponding quotient-groups with regard to $\Gamma$ are the dihedral groups of order $2n$. A set of generators is obtained for each and the fundamental regions are constructed. It has not been possible as yet to determine the arithmetic character of the substitutions of these subgroups; but a convenient criterion is found which completely characterizes the substitutions of each (§ 3). The finite group obtained by reducing the coefficients of all the substitutions of $\Gamma$ modulo $n$, is briefly discussed when $n$ is an odd prime $q$; two cases present themselves according as 2 is a quadratic residue, or non-residue of $q$ (§ 4).

Part III treats of the functions of $z$ belonging to the group $\Gamma$. RITTER's automorphic forms when regarded as functions of the ratio of the homogeneous variables give rise to the $\Theta$-functions, which are a simple generalization of POINCARÉ's theta-fuchsian functions (§ 1). With the aid of the hyperelliptic ($p = 2$) theta-constants it is possible to construct three functions $\theta, \phi, \psi$, by means of which every $\Theta$-function of $\Gamma$ can be rationally expressed (§ 5). Certain theorems concerning the expression of any theta-fuchsian function of degree greater than four as a product of functions of smaller degrees are deduced (§ 2). A principal automorphic function of $\Gamma$ is constructed (§ 4) and shown to be a simple function of the cross-ratio of the branch-points of the surface $R$. The paper closes (§ 6) with the derivation of certain infinite series. Exponential series are derived for $\theta, \phi, \psi$, and series similar to the Poincaré series, but more convenient, for $\phi^2$, $\psi$, and $\phi\psi$.

## PART I.

*The monodromies of the Riemann Surface $R$ defined by the equation*

$$y^4 = (x-\kappa_1)^2(x-\kappa_2)^2(x-\kappa_3)(x-\kappa_4)^3.$$

1. The surface $R$ is of genus $p = 2$. Two independent integrals of the first kind on $R$ are

$$v_1 = \int \frac{dx}{y}, \qquad v_2 = \int \frac{dx}{y'}, \tag{1}$$

where $y'$ is obtained from $y$ by interchanging $\kappa_3$ and $\kappa_4$. If we choose a system of cross-cuts $a_i, b_i$ $(i = 1, 2)$ as indicated in Fig. 1, and denote the moduli of periodicity as usual by $A_{ik}, B_{ik}$ $(i, k = 1, 2)$, the table of periods takes the following form

* *Elliptische Modulfunctionen*, vol. 1, pp 163–491.

| | $a_1$ | $a_2$ | $b_1$ | $b_2$ |
|---|---|---|---|---|
| $v_1$ | $A_{11}$ | $-iA_{11}$ | $B_{11}$ | $-A_{11}-iB_{11}$ |
| $v_2$ | $A_{12}$ | $iA_{12}$ | $B_{12}$ | $-A_{12}+iB_{12}$ |

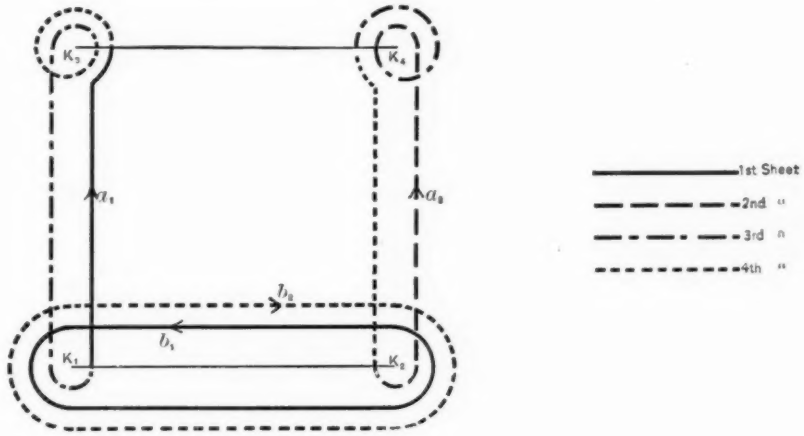


FIG. 1.

2. If we calculate the periods of two normal integrals $u_1$, $u_2$ of the first kind and if for $B_{11}/A_{11}$ we write $z/\sqrt{2}+(1+i)/2$ $(i^2=-1)$, we may, by making use of the bilinear relation

$$A_{11}B_{12}-B_{11}A_{12}+iA_{11}A_{12}=0,$$

put the table of periods for $u_1$, $u_2$ in the form

(2)

| | $a_1$ | $a_2$ | $b_1$ | $b_2$ |
|---|---|---|---|---|
| $u_1$ | $\pi i$ | 0 | $\pi i\left(\frac{z}{\sqrt{2}}+\frac{1}{2}\right)$ | $-\frac{\pi i}{2}$ |
| $u_2$ | 0 | $\pi i$ | $-\frac{\pi i}{2}$ | $\pi i\left(\frac{z}{\sqrt{2}}+\frac{1}{2}\right)$ |

3. Any continuous deformation of $R$ whereby the branch-points are caused to move in closed paths we call a monodromy of the branch-points.* Every monodromy of $R$ can be generated by the repeated application of the following

* Cf. BURKHARDT, *Systematik der hyperelliptischen Functionen*, Mathematische Annalen, vol. 35 (1890), p. 212, for nomenclature.

two: *a*) The interchange of $\kappa_1$ and $\kappa_2$, and *b*) The movement of $\kappa_2$ about $\kappa_3$.* Denote the resulting substitutions on $z$ by $S$ and $T_1$, respectively. They are

$$S(z) = z + \sqrt{2}, \qquad T_1(z) = \frac{-\sqrt{2}z - 1}{z}.\dagger$$

Introducing in the place of $T_1$ the substitution $T = ST_1$, we have as generators of the group $\Gamma$,

$$S(z) = z + \sqrt{2},$$

$$T(z) = -\frac{1}{z}.$$

## Part II.

### *The group* $\Gamma$ *of sign* $(0, 3; 2, 4, \infty)$.

The group $\Gamma$ generated by the two substitutions $S$ and $T$ of the last paragraph, will now be discussed in some detail. The treatment will follow closely that of the modular group in Fricke-Klein's *Theorie der Elliptischen Modulfunctionen*, vol. I, to which reference will be made by the letter M.

### §1. *The arithmetic character of the substitutions in* $\Gamma$. — *Fundamental region.*

1. *The group* $\Gamma$ *consists of the totality of substitutions,* $V$, *of the two types*

$$V' = \begin{vmatrix} \alpha & \beta\sqrt{2} \\ \gamma\sqrt{2} & \delta \end{vmatrix} \qquad \text{and} \qquad V'' = \begin{vmatrix} \alpha'\sqrt{2} & \beta' \\ \gamma' & \delta'\sqrt{2} \end{vmatrix}$$

*of determinant unity, in which* $\alpha, \beta, \gamma, \delta$ *and* $\alpha', \beta', \gamma', \delta'$ *are integers.*

That every substitution of $\Gamma$ is of one of the above types is easily seen. That every substitution $V$ is in $\Gamma$ is shown by proving that any $V$ may be reduced to the identical substitution by successive multiplication by $T$ and suitable powers of $S$. The reduction requires an even or an odd number of multiplications by $T$ according as $V$ belongs to the type $V'$ or $V''$; we call the substitutions of these types even and odd respectively.

2. The following kinds of substitutions occur in $\Gamma$.

*a*) The even substitutions:

i) elliptic, of period 2, when $\alpha + \delta = 0$,

ii) parabolic, when $\alpha + \delta = 2$,

iii) hyperbolic, when $\alpha + \delta > 2$.

*b*) The odd substitutions:

i) elliptic, of period 2, when $\alpha' + \delta' = 0$,

* Two monodromies are not regarded as distinct if they produce the same change in $z$.

† For a more detailed account of an analogous problem, see Hutchinson, *On a class*, etc., Transactions, vol. 3 (1902), p. 3.

ii) elliptic, of period 4, when $\alpha' + \delta = 1$,

iii) hyperbolic, when $\alpha' + \delta' > 1$.

3. A fundamental region for $\Gamma$ in the $z = z' + iz''$ plane is the infinite strip included between the two straight lines $z' = \pm \frac{1}{2}\sqrt{2}$ and exterior to the circle with unit radius about the origin. The points $z = i$, $z = \rho = (-1+i)/\sqrt{2}$, and $z = i\infty$ are fixed points of the substitutions $T$, $U = S^{-1}T$, and $S$, of which the first two are subject to the relations $T^2 = 1$, $U^4 = 1$. (See the region marked 1 in Fig. 2.)

## § 2. *Sets of conjugate substitutions in* $\Gamma$.

4. *Elliptic substitutions of period* 4 in $\Gamma$ are of two classes, according as the revolution about one of their fixed points due to a single application of the substitution is $\pi/2$ or $3\pi/2$. As in M. p. 263, we show that *all substitutions of period* 4 *of the former class are conjugate with* $U$; *all of the latter with* $U^3$; *and no substitution of the former is conjugate with any substitution of the latter.*

Further, *all even substitutions of period* 2 *are conjugate with* $U^2$ *and all odd substitutions of period* 2 *with* $T$.

5. Any parabolic substitution

$$S' = \begin{vmatrix} \alpha & \beta\sqrt{2} \\ \gamma\sqrt{2} & 2-\alpha \end{vmatrix}$$

may be written in the form

$$S' = \begin{vmatrix} 1+kcd & \frac{kd^2}{2}\sqrt{2} \\ -kc^2d\sqrt{2} & 1-kcd \end{vmatrix},$$

where $c$, $d$ are prime to each other and such that $(\alpha-1)/\gamma = -d/c$, and where $k$ is an integer. Either $k$ or $d$ is even. We define the *amplitude* of $S'$ as follows:

If $d$ is even, the amplitude is $k$; if $d$ is odd, the amplitude is $\frac{1}{2}k$.

With this definition we may show that *two parabolic substitutions of* $\Gamma$ *are conjugate if, and only if, they have the same amplitude* (M. pp. 264, 265).

For, if $d$ is even, we may determine two numbers $a$, $b$ such that $ad - bc = 1$, and if we place

$$W = \begin{vmatrix} a\sqrt{2} & b \\ c & \frac{d}{2}\sqrt{2} \end{vmatrix},$$

we have

$$WS'W^{-1} = \begin{vmatrix} 1 & k\sqrt{2} \\ 0 & 1 \end{vmatrix} = S^k.$$

If $d$ is odd we may determine two numbers $a'$, $b'$ such that $a'd - 2b'c = 1$, and if we write

$$W' = \begin{vmatrix} a' & b'\sqrt{2} \\ c\sqrt{2} & d \end{vmatrix},$$

we have

$$W' S' W'^{-1} = \begin{vmatrix} 1 & \frac{k}{2}\sqrt{2} \\ 0 & 1 \end{vmatrix} = S^{\frac{1}{2}k}.$$

6. No use will be made in the following of the conditions for the conjugacy of two hyperbolic substitutions. Their derivation, involving a discussion of a certain class of binary quadratic forms, may therefore be omitted.

## § 3. *The invariant subgroups of* $\Gamma$.

7. In discussing the subgroups of $\Gamma$ we confine ourselves to those which are invariant (self-conjugate) and assume familiarity with the general methods employed in M. pp. 305 ff. An invariant subgroup of $\Gamma$ of index $\mu$ and genus $p$ we denote by $\Gamma_{\mu,p}$; the fundamental region of the subgroup consisting of $\mu$ fundamental triangles,—or the corresponding closed surface,—by $F_{\mu,p}$. The outer edges of the triangles correspond in pairs and the substitutions of $\Gamma$ which transform two corresponding edges one into the other, form a set of generators of $\Gamma_{\mu,p}$.

The genus of an invariant subgroup $\Gamma_{\mu,p}$ is given by the equation (M. p. 341):

$$p = -\mu + 1 + \frac{\mu}{n_i}\cdot\frac{n_i - 1}{2} + \frac{\mu}{n_\rho}\cdot\frac{n_\rho - 1}{2} + \frac{\mu}{n_\infty}\cdot\frac{n_\infty - 1}{2},$$

where $2n_i$, $2n_\rho$, $2n_\infty$ are respectively the numbers of elementary triangles surrounding the points $i$, $\rho = (-1+i)/\sqrt{2}$, $i\infty$, or their equivalents on $F_{\mu,p}$. Clearly, $n_i$ is either 1 or 2, and $n_\rho$ either 1, 2, or 4; the symbol $\{n_i, n_\rho, n_\infty\}$ we call the *characteristic* of the subgroup.

A consideration of the above equation gives the following possible solutions:

i) $p = 0$, $\mu = 0$, characteristic: $\{1, 1, 1\}$,
$\mu = 2$, " $\{2, 1, 2\}$,
$\mu = 2$, " $\{1, 2, 2\}$,
$\mu = 4$, " $\{1, 4, 4\}$,
$\mu = 2n$, " $\{2, 2, n\}$ $(n = 1, 2, 3, \cdots, \infty)$,
$\mu = 8$, " $\{2, 4, 2\}$,
$\mu = 24$, " $\{2, 4, 3\}$,

ii) $p = 1$, $\mu$ is indeterminate, characteristic $\{2, 4, 4\}$,

iii) $p > 1$, $\mu$ is determinate, " $\{2, 4, n\}$.

The first solution leads to $\Gamma$ itself. For the next three it is easy to construct fundamental regions to show that corresponding subgroups exist. In all the other solutions the characteristic is either of the form $\{2, 2, n\}$ or $\{2, 4, n\}$ ($n$ a positive integer), and for the former the genus of the corresponding subgroup, if it exists, is necessarily $p = 0$, and the index $\mu = 2n$.

8. We denote by "the division $(2, 2, n)$" the division of the entire plane into congruent triangles whose angles are $\pi/2, \pi/2, \pi/n$; this is the familiar dihedral division. Now, as in M. p. 356, we may establish a correspondence between the division $(2, 2, n)$ and the division of the plane defined by $\Gamma$. Moreover this correspondence will define an invariant subgroup of $\Gamma$ for every positive integral value of $n$ (M. pp. 344–352). Hence

*Every division* $(2, 2, n)$ *defines an invariant subgroup* $\Gamma_{2n,0}$ *of* $\Gamma$, *of index* $2n$ *and genus zero. The corresponding quotient-group* $\Gamma/\Gamma_{2n,0} = G_{2n}$ *is holomorphic* * *with the dihedral group of order* $2n$.

This establishes the existence in $\Gamma$ of *an infinite series of invariant subgroups of genus zero*, a result in marked contrast to the modular group.

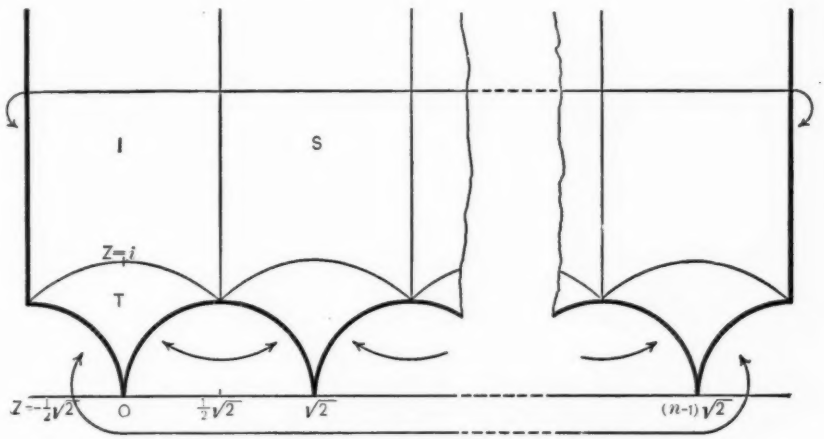


FIG. 2.

9. The accompanying figure (Fig. 2) gives a fundamental region for $\Gamma_{2n,}$ which is easily obtained from the correspondence above noted. The correspondence of the edges shows that of the $n + 1$ generators thus defined, $n - 1$ are

* "Holomorphic" is used for "holoedrically isomorphic."

elliptic of period two and one is parabolic of amplitude $n$. In fact, the generators are at once seen to be:

$$(ST)^2,\ S(ST)^2S^{-1},\ S^2(ST)^2S^{-2},\ \cdots,\ S^{n-2}(ST)^2S^{-n+2},\ S^n,\ S^n\cdot(S^{-1}T)^2.$$

The last of these may evidently be replaced by $(S^{-1}T)^2 = U^2$, so that $\Gamma_{2n,0}$ *is generated by n even substitutions of period two and one parabolic substitution of amplitude n*. Since $\Gamma_{2n,0}$ is invariant, it follows by Arts. 4, 5 that $\Gamma_{2n,0}$ *contains all the even substitutions of period two and all the parabolic substitutions of amplitude n*.

This follows also from a consideration of the correspondence established between the division $(2, 2, n)$ and the $\Gamma$-division; in fact it follows without any reference to the correspondence of sides of $F_{2n,0}$, that $\Gamma_{2n,0}$ is generated by the totality of substitutions conjugate with $U^2$ and $S^n$.

10. That all substitutions of $\Gamma_{2n,0}$ are even, is evident. But the determination of the arithmetic character of the substitutions of $\Gamma_{2n,0}$ offers as yet insurmountable difficulties of a number-theoretic character. It is however possible to give a simple criterion by which to determine whether or not a substitution is in $\Gamma_{2n,0}$.

Elementary group-theoretic considerations show that *in the isomorphism of* $\Gamma$ *with* $G_{2n}$, *the even substitutions of* $\Gamma$ *correspond to the cyclic subgroup* $H_n$ *of* $G_{2n}$, *of order n; while the odd substitutions of* $\Gamma$ *correspond to the remaining operations of* $G_{2n}$.* Now any even substitution of $\Gamma$ is of the form

$$V' = S^{a_1}\cdot TS^{a_2}T\cdot S^{a_3}\cdot TS^{a_4}T\cdot\ \cdots.$$

Since $S$ corresponds to an operation $s$ of period $n$ in $G_{2n}$, $V'$ will correspond to $s^{a_1-a_2+a_3-a_4+\cdots}$, which is identity only if the exponent is divisible by $n$. Hence $V'$ *is a substitution of* $\Gamma_{2n,0}$ *if, and only if,*

$$a_1 - a_2 + a_3 - a_4 + \cdots \equiv 0 \pmod{n}.$$

Any substitution $V'$ can be written as a continued fraction:

$$V'(z) = a_1\sqrt{2} + \cfrac{1}{-a_2\sqrt{2} - \cfrac{1}{-a_3\sqrt{2} - \cfrac{1}{-a_4\sqrt{2} - \cdots - \cfrac{1}{-(a_r\sqrt{2}+z)}}}},$$

where $r$ is odd. If the sum of the successive quotients $a_i\sqrt{2}$ taken alternately with a positive and negative sign is a multiple of $n\sqrt{2}$, then and only then will $V'$ belong to $\Gamma_{2n,0}$.

11. Similarly the division $(2, 4, n)$ defines an invariant subgroup $\Gamma_{\{n\}}$, whose index $\{n\}$ is infinite when $n > 3$; for the smaller values of $n$, we have $\{2\} = 8$ and $\{3\} = 24$. The subgroup $\Gamma_{\{n\}}$ is generated by the totality of parabolic substitutions of amplitude $n$ (M. p. 357–359); these have the form:

* The latter are all of period two.

$$S_k^n = \begin{vmatrix} 1 + ncd & \frac{1}{2}nd^2\sqrt{2} \\ -nc^2\sqrt{2} & 1 - ncd \end{vmatrix} \qquad [d \text{ even},$$

$$S_{k'}^n = \begin{vmatrix} 1 + 2ncd & nd^2\sqrt{2} \\ -2nc^2\sqrt{2} & 1 - 2ncd \end{vmatrix} \qquad [d \text{ odd},$$

where $c$, $d$ take all integral values prime to each other.

That these subgroups $\Gamma_{\{n\}}$ play the same important rôle in a systematic discussion of the subgroups of $\Gamma$ as do the corresponding subgroups in the modular group (M. p. 360 ff.), need hardly be mentioned. As in the latter group, so here the discussion of all the subgroups of a given "class" $n$ reduces itself to the investigation of the quotient-group $\Gamma/\Gamma_{\{n\}} = G_{\{n\}}$. This group $G_{\{n\}}$ is itself of infinite order, if $n > 3$, but, since $\Gamma_{\{n\}}$ is an invariant subgroup of $\Gamma_{2n,0}$, it follows that $G_{\{n\}}$ is isomorphic with the dihedral group $G_{2n}$. This gives an insight into the structure of the groups $G_{\{n\}}$, which might be of value in a further investigation.

12. *The subgroup $\Gamma_{\{n\}}$ contains no elliptic substitutions;* for if it did it would contain all conjugate to $U^2$, and then $\Gamma_{\{n\}}$ and $\Gamma_{2n,0}$ would be identical.* Also *the invariant subgroup $\Gamma_{\infty,0}$ of $\Gamma$, generated by all the conjugates of $U^2$, contains no parabolic substitutions;* for if it contained one of amplitude $n$ it would contain all, so that the conjugates of $U^2$ would generate $\Gamma_{2n,0}$; this is not possible. Now $\Gamma_{\infty,0}$ and $\Gamma_{\{n\}}$ must have substitutions in common; for otherwise every substitution of one would be commutative with every substitution of the other, which is easily shown not to be the case. These common substitutions form *an invariant subgroup of $\Gamma$, all of whose substitutions (except identity) are hyperbolic.*

## §4. *The congruence-groups of $\Gamma$ and the corresponding finite groups.*

13. Any subgroup of $\Gamma$, the arithmetic character of which may be completely defined by congruences, we call a congruence-group of $\Gamma$. All the substitutions of $\Gamma$ which are congruent to $\left|\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right|$ (mod. $n$), form an invariant subgroup $\Gamma_{\mu(n)}$ of $\Gamma$, which we call the principal congruence-group (mod. $n$). The index $\mu(n)$ of $\Gamma_{\mu(n)}$ is finite and is easily calculated (M. p. 395).

An even substitution can not be equivalent to an odd substitution; but it must now be noted that *even and odd substitutions may give rise to the same*

* The case $n = 1$ appears to offer an exception, since the product of the two parabolic substitutions $S$ and $TST$ gives an elliptic substitution. But $\Gamma_{\{n\}}$ does not exist (the characteristic (2, 4, 1) is impossible) and the totality of parabolic substitutions of amplitude one does generate $\Gamma_{2,0}$ of characteristic (2, 2, 1).

*reduced (mod. n) substitution, for certain values of n.* In fact the supposition that

$$V' = \begin{vmatrix} \alpha & \beta\sqrt{2} \\ \gamma\sqrt{2} & \delta \end{vmatrix}$$

and

$$V'' = \begin{vmatrix} \alpha'\sqrt{2} & \beta' \\ \gamma' & \delta'\sqrt{2} \end{vmatrix}$$

give rise to the same reduced substitution leads at once to the relations

$$\alpha \equiv \sigma\alpha'\sqrt{2}, \qquad \beta\sqrt{2} \equiv \sigma\beta', \qquad \gamma\sqrt{2} \equiv \sigma\gamma', \qquad \delta \equiv \sigma\delta'\sqrt{2} \pmod{n},$$

where the factor of proportionality $\sigma$ is of the form $\xi\sqrt{2}$ ($\xi$ an integer). The relation $\sigma^2 \equiv 1 \pmod{n}$, leads to the congruence

$$2\xi^2 \equiv 1 \pmod{n}$$

and if this is possible, then to every reduced substitution correspond both even and odd substitutions of $\Gamma$. Hence, *if the congruence* $2\xi^2 \equiv 1$ *(mod. n) is possible,* $\Gamma_{\mu(n)}$ *contains both even and odd substitutions; if this congruence is impossible,* $\Gamma_{\mu(n)}$ *contains only even substitutions.*

14. The quotient-group $\Gamma/\Gamma_{\mu(n)} = G_{\mu(n)}$ is obtained by reducing all the substitutions of $\Gamma$ (mod. $n$). In case $n$ is an odd prime $q$, $G_{\mu(n)}$ is easily determined. If we transform the subgroup $\Gamma_{2,0}$ consisting of all the even substitutions of $\Gamma$ by the substitution $z' = z\sqrt{2}$, we obtain the totality of substitutions $\left|\begin{smallmatrix} \alpha & \beta \\ 2\gamma & \delta \end{smallmatrix}\right|$; i. e., a subgroup of the modular group. If these substitutions be reduced (mod. $q$), we obtain the whole reduced modular group $G_{\frac{1}{2}q(q^2-1)}$, whose properties are well known (M. pp. 419–490). Now, since the congruence $2\xi^2 \equiv 1$ (mod. $q$) is possible only when $q$ is of the form $8h \pm 1$, we have:

*If q is a prime of the form* $8h \pm 1$, $G_{\mu(q)}$ *is holomorphic with the modular* $G_{\frac{1}{2}q(q^2-1)}$; *if q is of the form* $8h \pm 3$, $G_{\mu(q)}$ *is of order* $q(q^2-1)$ *and contains a subgroup of index 2 holomorphic with the modular* $G_{\frac{1}{2}q(q^2-1)}$.*

## PART III.

### *The functions associated with* $\Gamma$.

### § 1.—*The* $\Theta$*-functions.*

Let $\Theta(z)$ be a uniform function satisfying the relation

$$\Theta(z_k) = \mu_k(c_k z + d_k)^d \Theta(z), \tag{1}$$

* A similar result has been obtained by FRICKE, for a whole series of groups, of which, however, our $\Gamma$ is not a member.—Mathematische Annalen, vol. 39 (1891), p. 85.

where $z_k = (a_k z + b_k)/(c_k z + d_k)$ is a substitution with determinant unity, $\mu_k$ a constant, and $d$ an integer. In case of an elliptic substitution, $\mu_k$ must be of the form

$$\mu_k = e^{\frac{2\pi i \nu_k}{l_k}} \qquad \text{or} \qquad \mu_k = e^{\frac{\pi i}{l_k}(2\nu_k+1)} \qquad (\nu_k = 0, 1, 2, \cdots, l_k - 1)$$

according as $d$ is even or odd,* where $l_k$ is the period of the substitution in question. Let the substitution be

$$t_k = e^{\frac{2\pi i}{l_k}} t,$$

where $t = (z - z^{(1)})/(z - z^{(2)})$ and $t_k = (z_k - z^{(1)})/(z_k - z^{(2)})$.

i) If $d$ is even, (1) may be written

$$\Theta(z_k)(z_k - z^{(2)})^d = e^{\frac{\pi i(2\nu_k - d)}{l_k}} \Theta(z)(z - z^{(2)})^d, \tag{2}$$

and if we put

$$\Theta(z)(z - z^{(2)})^d = \sum_{p=0}^{\infty} A_p t^p,$$

by substituting the value of $t_k$ in terms of $t$, we obtain from (2) $A_p = 0$, except when $p \equiv \nu_k - \frac{1}{2}d \pmod{l_k}$. Hence, $\Theta(z)$ *vanishes at* $z = z^{(1)}$, *except when* $\nu_k - \frac{1}{2}d \equiv 0 \pmod{l_k}$.

ii) Similarly if $d$ is odd, we find $A_p = 0$, except when $p \equiv \nu_k - \frac{1}{2}(d-1) \pmod{l_k}$, and hence $\Theta(z)$ *vanishes at* $z = z^{(1)}$, *except when* $\nu_k - \frac{1}{2}(d-1) \equiv 0 \pmod{l_k}$.†

Applying these results to our group $\Gamma$, we seek first the difference between the number $r$ of zeros and the number $q$ of poles which lie inside the fundamental region $F$ of $\Gamma$. We have

$$2\pi i(r - q) = \int_{(F)} d \log \Theta(z),$$

the integral being taken over the entire boundary of $F$, the vertices $z = \rho$, $\rho_1 (= (1+i)/\sqrt{2})$, $i$, $i\infty$ being excluded by small circular arcs. In evaluating this integral we make use of the relations

$$\Theta(z + \sqrt{2}) = e^{\frac{\pi i s}{4}} \Theta(z) \qquad \begin{pmatrix} s = 0, 2, 4, 6 \text{ where } d \text{ is even} \\ s = 1, 3, 5, 7 \quad " \quad d \text{ is odd} \end{pmatrix},$$

$$\Theta\left(-\frac{1}{z}\right) = e^{\frac{\pi i t}{2}} z^d \Theta(z) \qquad \begin{pmatrix} t = 0, 2 \text{ where } d \text{ is even} \\ t = 1, 3 \quad " \quad d \text{ is odd} \end{pmatrix},$$

and of the expansions

* FRICKE-KLEIN, *Automorphe Functionen*, vol. 2, p. 85.

† For special case of the above ($\nu_k = 0$), cf. POINCARÉ, Acta Mathematica, vol. 1 (1882), pp. 218, 219.

$\Theta(z) = \left(\frac{z-\rho}{z-\rho'}\right)^{p_0} P_1\left(\frac{z-\rho}{z-\rho'}\right)$, where $P_1$ does not vanish when $z = \rho$,

$\Theta(z) = \left(\frac{z-i}{z+i}\right)^{p_0'} P_2\left(\frac{z-i}{z+i}\right)$, " $P_2$ " " " " $z = i$,

$\Theta(z) = e^{\frac{\pi i s z}{\nu\sqrt{2}}} P_3\left(e^{\frac{2\pi i z}{\nu\sqrt{2}}}\right)$, " $P_3$ " " " " $z = i\infty$.

Here $p_0$ and $p_0'$ are integers satisfying the congruences

$$p_0 \equiv t - \tfrac{1}{2}(s+d) \pmod 4,$$
$$p_0' \equiv \tfrac{1}{2}(t-d) \pmod 2;$$

these relations are readily obtained from those on the preceding page, by observing that for $p_0$ we have,

$$v_k = t - \tfrac{1}{2}s \qquad \text{or} \qquad v_k = t - \tfrac{1}{2}(s+1),$$

according as $d$ is even or odd, since the substitution in question is $S^{-1}T$; and for $p_0'$ we have

$$v_k = \tfrac{1}{2}t \qquad \text{or} \qquad v_k = \tfrac{1}{2}(t-1),$$

according as $d$ is even or odd.

With these specifications the above integral may be evaluated without difficulty. We obtain, then:

*The difference between the number of zeros and the number of poles of a* $\Theta$*-function belonging to* $\Gamma$, *which lie inside the region* $F$, *is given by*

$$r - q = \tfrac{1}{8}(d - 2p_0 - 4p_0' - s).$$

## § 2. *The theta-fuchsian functions of Poincaré.*

POINCARÉ's theta-fuchsian functions* are the $\Theta$-functions above defined for the particular values $\mu_k = 1$, i. e., $s = t = 0$. In this case $d$ is necessarily even, equal to $2m$ say; $m$ is called the *degrée* of the function. The zeros common to all theta-fuchsian functions of a given degree, are the *fixed* zeros; the others, the *movable* zeros.

Let $\Theta^{(m)}$ denote a theta-fuchsian function of degree $m$, which does not become infinite anywhere in the region $F$; such a function we call an integral theta-fuchsian function. Now $\Theta^{(m)}$ vanishes at $z = \rho$ of order $p_0$ at least, where $p_0$ is the smallest integer satisfying the congruence $p_0 + m \equiv 0 \pmod 4$; and at $z = i$ of order $p_0'$ at least, where $p_0'$ is the smallest integer satisfying $p_0' + m \equiv 0 \pmod 2$. All other zeros are movable. If $\Theta^{(m)}$ vanishes at $z = \rho$ or $z = i$ of

* Acta Mathematica, vol. 1 (1882), p. 210.

an order higher than $p_0$ or $p_0'$ respectively, we shall say that one or more movable zeros have merged into the point in question. With these conventions the number of movable zeros of $\Theta^{(m)}$ is given by

$$r = \tfrac{1}{4}(m - p_0 - 2p_0'),$$

or if we place $m = 4k + k'$, then

$$r = k \qquad (k' = 0, 2, 3),$$

$$r = k - 1 \qquad (k' = 1).$$

Hence a $\Theta^{(1)}$ does not exist; a $\Theta^{(2)}$ has a single zero, of order 2, at $z = \rho$; a $\Theta^{(3)}$ has a zero of order 1 at $z = \rho$ and one of order 1 at $z = i$ and nowhere else. If we regard functions which differ only by a constant factor as not distinct, it follows that *two integral theta-fuchsian functions of the same degree and with the same zeros are identical.* For the quotient of two such functions, being an automorphic function with no zeros or infinities in $F$, is constant. In particular, there can exist only one $\Theta^{(2)}$ and one $\Theta^{(3)}$.

A $\Theta^{(4)}$ has a zero of order 1 at some point in $F$. If $\Theta_1^{(4)}$ and $\Theta_2^{(4)}$ are two functions which do not have the same zero, and if $a_1$ and $a_2$ are respectively their values at an arbitrary point $z = \alpha$ in $F$, then

$$\Theta_\alpha^{(4)} = a_2\Theta_1^{(4)} - a_1\Theta_2^{(4)}$$

is a $\Theta^{(4)}$ which vanishes at $z = \alpha$. Hence, since two distinct $\Theta^{(4)}$'s exist (cf. below), *a $\Theta^{(4)}$ exists with any assigned zero, and any $\Theta^{(4)}$ is linearly expressible in terms of two of them.*

Further $\Theta^{(2)}\Theta^{(3)}$ is the only $\Theta^{(5)}$. Now let $\Theta_a^{(4k)}$ be a $\Theta^{(4k)}$ with $k$ zeros $a_i (i = 1, 2, \cdots, k)$ and let $\Theta_{a_i}^{(4)}$ be the $\Theta^{(4)}$ which vanishes at $a_i$. Then

$$\frac{\Theta_a^{(4k)}}{\Theta_{a_1}^{(4)} \cdot \Theta_{a_2}^{(4)} \cdots \Theta_{a_k}^{(4)}}$$

is automorphic; and, since it has no zeros or infinities in $F$, it is a constant. Hence:

*Every integral theta-fuchsian function of degree $4k$ is expressible in one way as the product of $k$ functions of degree 4.*

*Similarly, every integral theta-fuchsian function of degree $4k+\alpha$ ($\alpha=2, 3$) is the product of $\Theta^{(\alpha)}$ and $k$ functions of degree 4; and every function of degree $4k + 1$ is the product of $\Theta^{(2)}\Theta^{(3)}$ and $k - 1$ functions of degree 4.*

## § 3. *Transformation of the theta constants.*

In order actually to construct functions of the kind we have been considering, we make use of the hyperelliptic ($p = 2$) theta-constants,* whose moduli are

* i. e., theta-functions with zero arguments considered as functions of the parameter $z$ entering the moduli.

given by the table (2) in Part I. These are uniform functions of $z = z' + iz''$, defined for all points such that $z'' > 0$. Our object is to determine the effect on these theta-constants when $z$ is transformed according to the substitutions of $\Gamma$. In KRAZER and PRYM's* notation, let $a_{jk}$ be the old moduli and $b_{jk}$ the new. Then for the substitution $S$, we have

$$a_{11} = a_{22} = \pi i\left(\frac{z}{\sqrt{2}} + \tfrac{1}{2}\right),$$

$$b_{11} = b_{22} = \pi i\left(\frac{z}{\sqrt{2}} + \tfrac{3}{2}\right),$$

$$a_{12} = a_{21} = b_{12} = b_{21} = -\frac{\pi i}{2}.$$

We must determine integers $\alpha_{jk}$, $\beta_{jk}$, $\gamma_{jk}$, $\delta_{jk}$ satisfying the relations

$$\sum_{\kappa} b_{\nu\kappa}\left[\alpha_{\kappa\mu}\pi i + \sum_{\lambda}\beta_{\kappa\lambda}a_{\mu\lambda}\right] = \pi i\left[\gamma_{\nu\mu}\pi i + \sum_{\lambda}\delta_{\nu\lambda}a_{\mu\lambda}\right] \quad (\kappa, \lambda, \mu, \nu = 1, 2),$$

and certain bilinear relations.† By substituting the values of $\alpha_{jk}$ and $\beta_{jk}$, and equating like powers of $z$, we easily find the most general values for $\alpha_{jk}$, $\beta_{jk}$, $\gamma_{jk}$, $\delta_{jk}$. A particular set of values is the following:

$$\alpha_{11} = \alpha_{22} = \gamma_{11} = \gamma_{22} = \delta_{11} = \delta_{22} = 1,$$

$$\beta_{jk} = \alpha_{12} = \alpha_{21} = \gamma_{12} = \gamma_{21} = \delta_{12} = \delta_{21} = 0.$$

If, then, we denote by $\vartheta\left[\begin{smallmatrix} g_1 & g_2 \\ h_1 & h_2 \end{smallmatrix}\right](z)$ a theta-function with characteristic $g_i$, $h_i$, arguments zero and moduli $a_{ik}$, this set of values gives the following formula:

$$\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}(z + \quad \overline{2}) = e^{-\pi i(g_1^2 + g_2^2 - g_1 - g_2)}\,\vartheta\begin{bmatrix} g_1 & g_2 \\ g_1 + h_1 - \frac{1}{2} & g_2 + h_2 - \frac{1}{2} \end{bmatrix}(z).$$

Similarly, we obtain for the substitution $T$, if we place

$$b_{11} = b_{22} = \pi i\left(\tfrac{1}{2} - \frac{1}{z\sqrt{2}}\right),$$

the transformation represented by

$$\alpha_{11} = \alpha_{12} = \gamma_{11} = \delta_{11} = \delta_{21} = 0,$$

$$\alpha_{22} = \beta_{11} = \beta_{12} = \beta_{21} = \gamma_{22} = \delta_{12} = 1, \qquad \alpha_{21} = \beta_{22} = \gamma_{12} = \gamma_{21} = \delta_{22} = -1;$$

and hence the corresponding formula

* *Neue Grundlagen*, etc.

† KRAZER und PRYM, loc. cit., p. 121: $T_1$ or $T_2$

$$\vartheta\begin{bmatrix}\hat{g}_1 & \hat{g}_2\\ \hat{h}_1 & \hat{h}_2\end{bmatrix}\left(-\frac{1}{z}\right) = ze^{-\pi i[(g_1-g_2)^2+2g_1h_1+4g_2h_2-2g_1h_2+2h_2^2-g_1+g_2-2h_1+\frac{1}{4}]}\,\vartheta\begin{bmatrix}g_1 & g_2\\ h_1 & h_2\end{bmatrix}(z),$$

where

$$\hat{g}_1 = -h_1 - h_2, \qquad \hat{g}_2 = g_2 - g_1 + h_2 - h_1 - 1,$$
$$\hat{h}_1 = g_2 + h_2 - \tfrac{1}{2}, \qquad \hat{h}_2 = g_1 - g_2 - h_2 - \tfrac{1}{2}.$$

Further by considering the transformation for which we have $b_{jk} = a_{jk}$, we obtain identical relations among the theta constants. The most general transformation of this kind is given by the following:

i) $\alpha_{11} = \alpha_{22} = \delta_{11} = \delta_{22} = m, \quad \alpha_{21} = \gamma_{22} = \delta_{21} = n, \quad \alpha_{12} = \gamma_{11} = \delta_{12} = -n,$
$$\beta_{jk} = \gamma_{12} = \gamma_{21} = 0;$$

ii) $\alpha_{11} = \gamma_{12} = \delta_{11} = m, \quad \alpha_{22} = \gamma_{21} = \delta_{22} = -m, \quad \alpha_{12} = \alpha_{21} = \delta_{12} = \delta_{21} = n,$
$$\beta_{jk} = \gamma_{11} = \gamma_{22} = 0;$$

where $m$, $n$ are integers satisfying the relation $m^2 + n^2 = 1$. By considering all possible cases of these transformations and confining our attention to half-integer characteristics, we obtain a series of identities, which are all deducible from the two following:

$$\vartheta\begin{bmatrix}g_1 & g_2\\ h_1 & h_2\end{bmatrix}(z) = \vartheta\begin{bmatrix}g_2 & g_1\\ h_2 & h_1\end{bmatrix}(z),$$

($g_i$, $h_i$ half-integers).

$$\vartheta\begin{bmatrix}g_1 & g_2\\ h_1 & h_2\end{bmatrix}(z) = e^{2g_1g_2\pi i}\,\vartheta\begin{bmatrix}g_1 & g_2\\ h_1-g_2, & g_1-h_2\end{bmatrix}(z)$$

These relations reduce the ten theta-constants with half-integer characteristics to five, which we designate as follows:*

$$\theta = \vartheta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](z),$$
$$\theta_1 = \vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](z), \qquad \theta_3 = \vartheta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](z) = \vartheta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](z),$$
$$\theta_2 = \vartheta\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right](z), \qquad \theta_4 = \vartheta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](z) = -i\vartheta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right](z).$$

From the general relations among the theta constants we obtain a set of identities, which may all be deduced from three, viz.:

$$\theta_3^2 + \theta_4^2 = \theta_1^2, \qquad \theta_2^2 + \theta_4^2 = \theta_3^2, \qquad \theta_3^2\theta_4^2 = \theta^4. \tag{3}$$

From the formulæ obtained above we may exhibit the effects of the substitutions $S$, $T$ on $\theta$, $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ in the following table:

*The denominators in the characteristics have been omitted.

| | $\theta$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ |
|---|---|---|---|---|---|
| $S$ | $e^{\pi i/4}\theta$ | $\theta_2$ | $\theta_1$ | $\theta_3$ | $i\theta_4$ |
| $T$ | $iz\theta$ | $e^{-\pi i/4}z\theta_2$ | $e^{-3\pi i/4}z\theta_1$ | $e^{-3\pi i/4}z\theta_4$ | $e^{-\pi i/4}z\theta_3$ |

We note at once that $\theta_1^2\theta_2^2$ is a theta-fuchsian function of degree 2, which accordingly vanishes at $z=\rho$ and at no other point of $F$. * It is the $\Theta^{(2)}$ of the preceding article. Also $\theta^8$ is a $\Theta^{(4)}$ which vanishes at $z=i\infty$. We have thus in $\theta_1^4\theta_2^4$ and $\theta^8$, two linearly independent $\Theta^{(1)}$'s.

## § 4. — *A principal automorphic function of* $\Gamma$.

The function

$$\zeta=\frac{\theta_1^4\theta_2^4}{\theta^8}$$

is an automorphic function which has a single zero and a single infinity in $F$; it is, therefore, a *principal* function.

*Every automorphic function of* $\Gamma$ *is rationally expressible in terms of* $\zeta$.† The Rosenhain moduli are‡

$$\kappa^2=\frac{\theta_4^2}{\theta_1^2},\qquad \lambda^2=\frac{\theta_4^2}{\theta_3^2},\qquad \mu^2=\frac{\theta^4}{\theta_1^2\theta_3^2}.$$

Hence from (3) we have

$$\mu^2=2\kappa^2,\qquad \frac{1}{\kappa^2}-\frac{1}{\lambda^2}=1. \tag{4}$$

We express the Rosenhain moduli in terms of the branch points $\kappa_1$, $\kappa_2$, $\kappa_3$, $\kappa_4$ of $R$ (Part I) by transforming the integrals (1) into the Rosenhain normal form by means of a birational transformation. The following will serve:

$$\xi=\sqrt{\frac{x-\kappa_3}{x-\kappa_4}}=\frac{y^2}{(x-\kappa_1)(x-\kappa_2)(x-\kappa_4)^2}.$$

We readily obtain

$$\frac{2}{\kappa_4-\kappa_3}\cdot\frac{(x-\kappa_4)^{\frac{1}{2}}d\xi}{(x-\kappa_3)^{\frac{1}{2}}(x-\kappa_1)^{\frac{1}{2}}(x-\kappa_2)^{\frac{1}{2}}}=\frac{dx}{y}.$$

If we place

$$\frac{(x-\kappa_1)^{\frac{1}{2}}(x-\kappa_2)^{\frac{1}{2}}(x-\kappa_3)^{\frac{1}{2}}}{(x-\kappa_4)^{\frac{1}{2}}}=\eta,$$

* In fact, $\theta_1$ vanishes at $z=e^{\pi i/4}$ and $\theta_2$ at $z=\rho=e^{3\pi i/4}$.

† FRICKE-KLEIN, loc. cit., vol. 2, pp. 19, 20.

‡ KRAUSE, *Transformation Hyperelliptischer Functionen*, p. 36.

we have

$$v_1 = \frac{2}{\kappa_4 - \kappa_3}\int \frac{\xi d\xi}{\eta}, \qquad v_2 = \frac{2}{\kappa_4 - \kappa_3}\int \frac{d\xi}{\eta}.$$

Calculating $\eta$ in terms of $\xi$, we have

$$(\kappa_4 - \kappa_3)^2\eta^2 = \xi\,[(\kappa_4 - \kappa_1)\xi^2 - (\kappa_3 - \kappa_1)][(\kappa_4 - \kappa_2)\xi^2 - (\kappa_3 - \kappa_2)].$$

The branch-points of $\eta(\xi)$ are given by the roots of $\eta = 0$; if we place

$$\sqrt{\frac{\kappa_3 - \kappa_1}{\kappa_4 - \kappa_1}} = M \qquad \text{and} \qquad \sqrt{\frac{\kappa_3 - \kappa_2}{\kappa_3 - \kappa_2}} = N,$$

they are

$$\xi_1 = 0, \qquad \xi_2 = M, \qquad \xi_3 = N, \qquad \xi_4 = \infty, \qquad \xi_5 = -M, \qquad \xi_6 = -N.$$

Then, we have,

$$\kappa^2 = \frac{\xi_1 - \xi_5}{\xi_2 - \xi_5} : \frac{\xi_1 - \xi_3}{\xi_2 - \xi_3} = (\xi_1\xi_5\xi_2\xi_3) = \frac{N - M}{2N},$$

$$\lambda^2 = (\xi_1\xi_6\xi_2\xi_3) = \frac{N - M}{N + M},$$

$$\mu^2 = (\xi_1\xi_4\xi_2\xi_3) = \frac{N - M}{N}.$$

These satisfy the relations (4).

Now, we have

$$\zeta(z) = \tfrac{1}{4}\frac{(1 - 2\kappa^2)^2}{\kappa^4(1 - \kappa^2)^2} = 4\,\frac{\left(\frac{M}{N}\right)^2}{\left(1 - \left(\frac{M}{N}\right)^2\right)^2},$$

and

$$\left(\frac{M}{N}\right)^2 = (\kappa_3\kappa_1\kappa_4\kappa_2) = \sigma \text{ (say)}.$$

Hence we obtain,

$$\zeta(z) = \frac{4\sigma}{(1 - \sigma)^2},$$

and therefore:

*Every automorphic function of* $\Gamma$ *is rationally expressible in terms of the cross-ratio* $\sigma$ *of the branch-points* $\kappa_1, \kappa_2, \kappa_3, \kappa_4$; *and every rational function of* $\sigma + 1/\sigma$ *is automorphic.*

## § 5. *The* $\Theta$*-functions of* $\Gamma$.

We denote the function $\theta_1\theta_2$ by $\phi$, and define a new function $\psi$ by the relation

$$\psi = \frac{\Theta^{(3)}}{\phi}.$$

The function $\psi$ has a single simple zero at $z = i$. In $\psi$, $\phi$, $\theta$, then, we have three functions which vanish at $i$, $\rho$, $i\infty$ respectively, and which have no other zeros in $F$. If $z$ be transformed according to any substitution in $\Gamma$, they remain unchanged except for certain factors; the latter are given in the following table:

| | $\theta$ | $\phi$ | $\psi$ |
|---|---|---|---|
| $S$ | $e^{\pi i/4}$ | 1 | 1 |
| $T$ | $iz$ | $-z^2$ | $-z^4$ |

If the arbitrary constant multiplier be so chosen that $\psi(i\infty) = 1$, then $\psi$ is given by the following equation:

$$\psi = \tfrac{1}{2}(\theta_1^4 + \theta_2^4).$$

Moreover from the relations (3) follows at once the identity

$$\theta^8 + \phi^4 - \psi^2 = 0.*$$

By means of $\phi$ and $\psi$ all theta-fuchsian functions of $\Gamma$ may be rationally expressed since $\phi^2$ is $\Theta^{(2)}$ and $\phi\psi$ is $\Theta^{(3)}$, and since $\phi^4$ and $\psi^2$ are two linearly independent $\Theta^{(4)}$'s. But further we may prove that

*Any $\Theta$-function belonging to $\Gamma$ of whatever dimension or multiplier-system is a rational function of $\theta$, $\phi$, $\psi$.*

The proof is simple; the following is a typical part. Denote by $\Theta_{s,t}$ a $\Theta$-function whose multipliers are given by $s$, $t$ (cf. Part. III, § 1). *The case* $s = 2$, $t = 0$: Here belong $\theta^2\phi$ and $\theta^2\psi$. Moreover, since here we have $p_0 \equiv -1 - m \pmod 4$ and $p_0' \equiv -m \pmod 2$ $(2m = d)$, every $\Theta_{2,0}$ vanishes either at $z = \rho$ or $z = i$; also it is $0^2$ at $z = i\infty$. Hence either $\Theta_{2,0}/\theta^2\phi$ or $\Theta_{2,0}/\theta^2\psi$ are theta-fuchsian. The treatment of all other possible combinations $s$, $t$ is similar. We find then:

*Every integral $\Theta$-function belonging to $\Gamma$ is of the form $\theta^\alpha\phi^\beta\psi^\gamma\Pi_i\Theta_i^{(4)}$ $(\alpha < 8, \beta < 2, \gamma < 2)$, the $\Theta_i^{(4)}$'s being integral theta-fuchsian functions of degree* 4.

The simplest $\Theta$-functions for each multiplier-system are given by the following table:

| | $s=0$ | $s=2$ | $s=4$ | $s=6$ | | $s=1$ | $s=3$ | $s=5$ | $s=7$ |
|---|---|---|---|---|---|---|---|---|---|
| $t=0$ | $\theta^8$, $\phi^2$, $\psi^2$ | $\theta^2\phi$, $\theta^2\psi$ | $\theta^4$ | $\theta^6\phi$, $\theta^6\psi$ | $t=1$ | $\theta$ | $\theta^3\phi$, $\theta^3\psi$ | $\theta^5$ | $\theta^7\phi$, $\theta^7\psi$ |
| $t=2$ | $\phi$, $\psi$ | $\theta^2$ | $\theta^4\phi$, $\theta^4\psi$ | $\theta^6$ | $t=3$ | $\theta\phi$, $\theta\psi$ | $\theta^3$ | $\theta^5\phi$, $\theta^5\psi$ | $\theta^7$ |

By analogy with the functions $\theta$, $\phi$, it might be supposed that $\psi$ could be represented as the product of four theta-constants $(p = 2)$. It is perhaps of

*cf. FRICKE-KLEIN, loc. cit., vol. 2, p. 75.

sufficient interest to show that this is impossible. In fact, if the characteristic of a constituent theta-constant be $\left[\begin{smallmatrix} g_1 & g_2 \\ h_1 & h_2 \end{smallmatrix}\right]$, it follows (by considering the theta-series) from the fact that $\psi$ does not vanish at $z = i\infty$ that $g_1, g_2$ must be integers; and since $T$ must transform each constituent into one of the same kind (i. e., with $g_1, g_2$ integers) it follows from the formula in Part III, § 3, that $h_1 + h_2$ and $h_1 - h_2$ must be integers. Hence, the constituent theta-constants in any such product would have to have half-integer characteristics. But these have already been fully discussed in regard to their behavior toward $S$ and $T$, and no combination of them into products will give $\psi$.

## § 6. *Infinite series for* $\theta, \phi, \psi$.

1) *Exponential series.*

The exponential series for $\vartheta\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right](z)$ readily takes the form

$$\theta = e^{\frac{\pi i}{8}} \sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} e^{(n_1-n_2)(n_1-n_2+1)\frac{\pi i}{2}} e^{[(2n_1+1)^2+4n_2^2]\frac{\pi i z}{4\sqrt{2}}}.$$

The first factor after the signs of summation equals $+1$ or $-1$ according as $n_1 - n_2 \equiv \frac{0}{3}$ or $n_1 - n_2 \equiv \frac{1}{2} \pmod 4$. The second factor is such that if $(a, b)$ are any two values of $(n_1, n_2)$, then the pairs $(a, \pm b)$ and $(-a-1, \pm b)$ give it the same values. By considering the numbers $n_1 - n_2$ in these four cases, it follows that all terms due to an odd $n_2$ cancel in pairs. We may then write the series for $\theta$ as follows:

$$e^{\frac{-\pi i}{8}}\theta = 2\sum_{n_1=0}^{\infty} \pm e^{(2n_1+1)^2\frac{\pi i z}{4\sqrt{2}}} + 4\sum_{n_1=0}^{\infty} \sum_{n_2=2,\,4,\,6,\,\ldots} \pm e^{[(2n_1+1)^2+4n_2^2]\frac{\pi i z}{4\sqrt{2}}}$$

where the upper sign is used whenever $n_1 - n_2 \equiv \frac{0}{3} \pmod 4$, the lower whenever $n_1 - n_2 \equiv \frac{1}{2} \pmod 4$. This gives

$$e^{\frac{-\pi i}{8}}\theta = 2e^{\frac{\pi i z}{4\sqrt{2}}} - 2e^{\frac{9\pi i z}{4\sqrt{2}}} - 4e^{\frac{17\pi i z}{4\sqrt{2}}} - 6e^{\frac{25\pi i z}{4\sqrt{2}}} - \cdots.$$

To obtain a series for $\phi = \theta_1\theta_2$, we proceed as follows. The series for $\theta_1 = \vartheta\left|\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right|(z)$ gives readily

$$\theta_1 = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} i^{\varepsilon_m} e^{(m_1^2+m_2^2)\frac{\pi i z}{\sqrt{2}}} \quad \left(\begin{array}{l} \varepsilon_m = 0, \text{ when } m_1 - m_2 \text{ is even} \\ \varepsilon_m = 1, \text{ when } m_1 - m_2 \text{ is odd} \end{array}\right), \tag{5}$$

and since $\theta_2(z) = \theta_1(z + \sqrt{2})$, we have

$$\theta_2 = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} i^{-\varepsilon_n} e^{(n_1^2+n_2^2)\frac{\pi i z}{\sqrt{2}}} \quad \left(\begin{array}{l} \varepsilon_n = 0, \text{ when } n_1 - n_2 \text{ is even} \\ \varepsilon_n = 1, \text{ when } n_1 - n_2 \text{ is odd} \end{array}\right). \tag{6}$$

Multiplying (5) and (6), we have

$$\phi = \sum_{-\infty}^{\infty} 4\, i^{\varepsilon_m - \varepsilon_n} e^{(m_1^2+m_2^2+n_1^2+n_2^2)\frac{\pi i z}{\sqrt{2}}},$$

where $m_1$, $m_2$, $n_1$, $n_2$ take independently all integral (and zero) values from $-\infty$ to $+\infty$. Since every integer may be represented as the sum of four squares, this series may be written in the form

$$\phi = \sum_{k=0}^{\infty} A_k e^{\frac{k\pi iz}{\sqrt{2}}}, \qquad A_k = \sum_{k} i^{\epsilon_m - \epsilon_n},$$

where the latter summation extends over all the representations of $k$ as the sum of four squares.

Since $\phi(z+\sqrt{2}) = \phi(z)$ it follows that $A_k$ vanishes when $k$ is odd; an even integer $k$ can be represented as the sum of the squares of four integers $m_1$, $m_2$, $n_1$, $n_2$ only when $m_1 - m_2$ and $n_1 - n_2$ are either both even or both odd, so that in any case $\epsilon_m - \epsilon_n = 0$. Hence when $k$ is even, $A_k$ is equal to the total number of representations of $k$ as the sum of four squares, i. e., $A_k$ is equal to 24 times the sum of all odd divisors of $k$.* If $k$ be written in the form

$$k = 2^{a_0} p_1^{a_1} p_2^{a_2} \cdots,$$

the sum of all odd divisors is

$$s(k) = \prod_i \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

With this notation, we have then

$$\phi = 1 + 24 \sum_{k} s(k) e^{\frac{k\pi iz}{\sqrt{2}}} \qquad (k = 2, 4, 6, \cdots, \infty).$$

This gives

$$\phi = 1 + 24e^{\frac{2\pi iz}{\sqrt{2}}} + 24e^{\frac{4\pi iz}{\sqrt{2}}} + 96e^{\frac{6\pi iz}{\sqrt{2}}} + \cdots.$$

A series for $\psi = \frac{1}{2}(\theta_1^2 + i\theta_2^2)(\theta_1^2 - i\theta_2^2)$ may be obtained in a similar manner. We have

$$\theta_1^2 + i\theta_2^2 = \sum{}^4 (i^{\epsilon_m + \epsilon_n} + i^{3\epsilon_m + 3\epsilon_n + 1}) e^{(m_1^2 + m_2^2 + n_1^2 + n_2^2)\frac{\pi iz}{\sqrt{2}}}$$

$$= \sum B_k e^{\frac{k\pi iz}{\sqrt{2}}},$$

suppose, where $\epsilon_m$, $\epsilon_n$ have the same significance as before. Let $t(k)$ be the number of representations of $k$ as the sum of four squares. If $k$ is odd, one of the numbers $\epsilon_m$, $\epsilon_n$ is zero and the other unity; hence we have

$$B_k = (1+i)t(k) \qquad (k \text{ odd}).$$

If $k$ is even we distinguish two cases:

* EISENSTEIN, Crelle's Journal, vol. 35 (1847), p. 133; BACHMAN, *Zahlentheorie*, vol. 4, part 1, p. 603.

1) if $k \equiv 0 \pmod 4$, we have $\epsilon_m = \epsilon_n = 0$, and

$$B_k = (1+i)t(k) \qquad (k \equiv 0 \pmod 4).$$

2) if $k \equiv 2 \pmod 4$, of the numbers $m_1$, $m_2$, $n_1$, $n_2$, two are even and two are odd. Now it is easy to show that in this case one-third of the total number of representations of $k$ give $\epsilon_m + \epsilon_n = 0$, and the remaining two-thirds give $\epsilon_m + \epsilon_n = 2$. For each of the former we obtain a coefficient $1+i$; for each of the latter a coefficient $-1-i$. Hence we have here

$$B_k = -\tfrac{1}{3}(1+i)t(k) \qquad (k \equiv 2 \pmod 4).$$

In general then, we have

$$B_k = \mu_k(1+i)t(k),$$

where

$$\mu_k = 1 \quad \text{if} \quad k \text{ is odd},$$
$$= 1 \quad \text{if} \quad k \equiv 0 \pmod 4,$$
$$= -\tfrac{1}{3} \quad \text{if} \quad k \equiv 2 \pmod 4.$$

Similar reasoning gives

$$\theta_1^2 - i\theta_2^2 = \sum B'_k e^{\frac{k\pi i z}{\sqrt{2}}},$$

where

$$B'_k = \nu_k(1-i)t(k),$$

in which

$$\nu_k = -1 \quad \text{if} \quad k \text{ is odd},$$
$$= +1 \quad \text{if} \quad k \equiv 0 \pmod 4,$$
$$= -\tfrac{1}{3} \quad \text{if} \quad k \equiv 2 \pmod 4.$$

Hence, finally, since $(1+i)(1-i) = 2$ we have

$$\psi = \sum_{k=0}^{\infty} C_k e^{\frac{k\pi i z}{\sqrt{2}}},$$

where

$$C_k = \sum_{s=0}^{k} \mu_s \nu_{k-s} t(s) t(k-s).$$

Transformation by $S$ shows that $C_k = 0$, when $k$ is odd. Hence we need consider only even values of $k$. The above formula gives readily

$$C_0 = 1, \quad C_2 = -80, \quad C_4 = -400, \quad C_6 = -2240, \quad \cdots.$$

2) *Series similar to Poincaré's.*

The series, derived by POINCARÉ * to represent his theta-fuchsian functions

* Acta Mathematica, loc. cit.

and subsequently generalized by RITTER* to apply to any Θ-function with whatever multiplier-system, are well known and may be used to represent the Θ-functions of Γ, provided the dimension is not less than four. However, the possibility of their identical vanishing is troublesome. This difficulty may be avoided by forming—in a way to be immediately set forth—series similar to the POINCARÉ-RITTER series, but which contain only a portion of the terms of the latter.

Let

$$z_j = f_j(z) \qquad (j = 0, 1, 2, \cdots, \infty)$$

represent the totality of substitutions of Γ which transform the fundamental region $F$ into a region lying between the straight lines $z' = \pm 1/\sqrt{2}\,(z = z' + iz'')$, and let

$$z_j = \frac{\alpha_j z + \beta_j \sqrt{2}}{\gamma_j \sqrt{2}\, z + \delta_j} \qquad \text{or} \qquad z_j = \frac{\alpha_j' \sqrt{2} z + \beta_j'}{\gamma_j' z + \delta_j' \sqrt{2}},$$

according as the $f_j$ is even or odd. Then, *of the different denominators of all the substitutions of* Γ *each occurs once and but once as a denominator of an* $f_j$. That every denominator occurs is seen from the fact that, if $V_k$ is any substitution of Γ, $S^n V_k$ is a substitution with the same denominator; and $n$ may always be so chosen that $S^n V_k$ is equal to an $f_j$. To show that each denominator occurs only once, observe that if two substitutions $f_j$ with the same denominator existed they would both transform the point $i$ into a point whose real part in absolute value is less than $1/\sqrt{2}$. If one of these, say $f_k$, be even, it transforms $i$ into

$$f_k(i) = \frac{\alpha_k i + \beta_k \sqrt{2}}{\gamma_k \sqrt{2}\, i + \delta_k},$$

of which the real part is

$$\left| \frac{\alpha_k \gamma_k + \beta_k \delta_k}{2\gamma_k^2 + \delta_k^2} \sqrt{2} \right| < \frac{1}{\sqrt{2}},$$

so that

$$|\alpha_k \gamma_k + \beta_k \delta_k| < \tfrac{1}{2}(2\gamma_k^2 + \delta_k^2).$$

Since the determinant of every substitution is unity, any other substitution $f_l$ with the same denominator as $f_k$ will have

$$\begin{aligned} \alpha_l &= \alpha_k + 2\gamma_k t, \\ \beta_l &= \beta_k + \delta_k t \end{aligned} \qquad (t \text{ an integer})$$

such that

$$|\alpha_l \lambda_k + \beta_l \delta_k| = |\alpha_k \gamma_k + \beta_k \delta_k + (2\gamma_k^2 + \delta_k^2) t| < \tfrac{1}{2}(2\gamma_k^2 + \delta_k^2);$$

* *Die eindeutigen automorphen Formen, etc.*, Mathematische Annalen, vol. 41 (1892), p. 1; cf. also FRICKE-KLEIN, loc. cit., vol. 2, pp. 137 ff.

this is impossible except when $t = 0$, or when $f_l = f_k$. Similar reasoning will apply to the case when $f_k$ is odd and leads to the same result.

We have then without difficulty:

$$(7) \qquad \sum_{j=0}^{\infty}\left[\frac{df_j(z)}{dz}\right]^m = \sum_k \frac{1}{(\gamma_k z + \delta_k)^{2m}} + \sum_k \frac{1}{(\gamma_k' z + \delta_k')^{2m}},$$

where $\gamma_k$, $\delta_k$ and $\gamma_k'$, $\delta_k'$ run through all *different* pairs of values occurring in the denominators of the substitutions of $\Gamma$. Moreover by comparison with the well-known series *

$$\sum \frac{1}{(mz+n)^{\nu}} \qquad (m, n \text{ integers}),$$

we see that *the series* (7) *converges uniformly and unconditionally for all values of* $z$ *which are not purely real, provided only* $m > 1$.

Now, *the function*

$$\Theta^{(m)} = \sum_j \left[\frac{df_j(z)}{dz}\right]^m$$

*is theta-fuchsian of degree* $m$.† For if $z_k = V_k(z)$ be any substitution of $\Gamma$, then we have

$$\Theta^{(m)}(z_k) = \sum_j \left[\frac{df_j(z_k)}{dz_k}\right]^m = \left(\frac{dz_k}{dz}\right)^{-m} \sum_j \left[\frac{df_j[V_k(z)]}{dz}\right]^m.$$

Now the series of denominators of $f_j V_k$ $(j = 0, 1, 2, \cdots)$ is the same as the series of denominators of $f_j$ (except possibly as regards the sign; and this difference is destroyed by the *even* exponent which affects all the denominators in the series). For, (1) all the denominators of the series $f_j$ occur in $f_j V_k$, since if $f_n$ be any $f_j$, $S^a$ may be so chosen that $S^a f_n V_k^{-1} = f_{n'}$; and $f_{n'} V_k$ has the same denominator as $f_n$; and (2) no denominator occurs more than once, since if $f_n V_k$ and $f_p V_k$ had the same denominator, multiplication by certain powers of $S$ would have to reduce them both to the same $f_j$; this would lead at once to a relation $S^a f_n = f_p$, which is impossible. Hence we have

$$\sum_j \left[\frac{df_j[V_k(z)]}{dz}\right]^m = \sum_j \left[\frac{df_j(z)}{dz}\right],$$

and, therefore, *the* $\Theta^{(m)}(z)$, *defined above, has the property*

$$\Theta^{(m)}(z_k) = \left(\frac{dz_k}{dz}\right)^{-m} \Theta^{(m)}(z),$$

*where* $z_k = V_k(z)$ *is any substitution of* $\Gamma$.

* EISENSTEIN, Crelle, vol. 35 (1847), p. 161.

† The use of such a series is suggested in FRICKE-KLEIN, loc. cit., vol. 2, p. 156.

Moreover, *the series* (7) *does not vanish identically*, for it evidently does not vanish for $z = i\infty$. *We have then, if we write*

$$\Sigma \frac{1}{(\gamma_k \sqrt{2} z + \delta_k)^{2m}} = \Sigma'_{2m}, \qquad \Sigma \frac{1}{(\gamma'_k z + \delta'_k \sqrt{2})^{2m}} = \Sigma''_{2m},$$

$$\Sigma'_{2m} + \Sigma''_{2m} = \Sigma_{2m},$$

*the following expansions:*

$$\Theta^{(2)} = \phi^2 = \Sigma_4,$$

$$\Theta^{(3)} = \phi\psi = \Sigma_6,$$

$$\psi = \Sigma'_4 - \Sigma''_4.$$

CORNELL UNIVERSITY,
*February*, 1903.

# ON THE DEFINITION OF REDUCIBLE HYPERCOMPLEX NUMBER SYSTEMS*

BY

SAUL EPSTEEN

According to PEIRCE† and SCHEFFERS‡ a hypercomplex number system is said to be reducible when, by a suitable choice of units

$$E \equiv E_j E_k \equiv e_1 \cdots e_m e_{m+1} \cdots e_n,$$

with the relations

$$e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3},$$

the following conditions are fulfilled:§

$C_1$) $E_j$ forms a system by itself, i. e.,

$$e_{j_1} e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3} \qquad (\gamma_{j_1 j_2 k} = 0);$$

$C_2$) $E_k$ forms a system by itself, i. e.,

$$e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3} \qquad (\gamma_{k_1 k_2 j} = 0);$$

$A$) $$e_j e_k = 0 \qquad (\gamma_{jkl} = 0);$$

$B$) $$e_k e_j = 0 \qquad (\gamma_{klj} = 0).$$

One of the chief results of this paper is that *for systems $E$ containing a modulus the conditions $C_1$, $C_2$ are unnecessary, being consequences of the others.* In other words, *a hypercomplex number system containing a modulus is reducible if:*

$$A)\ e_j e_k = 0 \qquad \textit{and} \qquad B)\ e_k e_j = 0.$$

* Presented to the Society (Chicago) January 1, 1904. Received for publication November 14, 1903.

† BENJ. PEIRCE, American Journal of Mathematics, vol. 4 (1881), p. 100.

‡ SCHEFFERS, Mathematische Annalen, vol. 39 (1891), p. 317.

§ In this paper the scheme of subscripts is the following:

$$i = 1, \cdots, n\,;\ j = 1, \cdots, m\,;\ k = m+1, \cdots, n.$$

## 1. *Preliminary Remarks.*

It is assumed that the hypercomplex number system $E \equiv e_1 \cdots e_n$ is associative, that is, that

$C_a$) $$(e_{i_1} e_{i_2}) e_{i_3} = e_{i_1} (e_{i_2} e_{i_3}).$$

Letting

$$X = \sum_{i_1} x_{i_1} e_{i_1}, \qquad Y = \sum_{i_2} y_{i_2} e_{i_2}$$

be two general numbers of the system, then we have

$$X' = XY = \sum_{i_3} x'_{i_3} e_{i_3} = \sum_{i_1 i_2} x_{i_1} y_{i_2} e_{i_1} e_{i_2}; \qquad x'_{i_3} = \sum_{i_1 i_2} \gamma_{i_1 i_2 i_3} x_{i_1} y_{i_2}.$$

The necessary and sufficient condition that the equation for $Y$,

$$0 = XY$$

$X$ being given), have a solution $Y \neq 0$, in addition to $Y = 0$, is that

$$\Delta_x \equiv \left| \sum_{i_1} \gamma_{i_1 i_2 i_3} x_{i_1} \right| = 0 \qquad (i_2, i_3 = 1, \cdots, n).$$

In this case $X$ is said to be a left hand divisor of zero. Further $\Delta_x \neq 0$ is the necessary and sufficient condition that in $X' = XY$ the left hand quotient $Y$ of any $X'$ by $X$ be existent (in fact uniquely).

Hence the fulfillment of the condition

$C_l$) $$\Delta_x \equiv \left| \sum_{i_1} \gamma_{i_1 i_2 i_3} x_{i_1} \right| \neq 0 \qquad (i_2, i_3 = 1, \cdots, n)$$

implies that

$C_l$) not every $X$ is a left hand divisor of zero.

Similarly, the fulfilment of the condition

$C_r$) $$\Delta'_y \equiv \left| \sum_{i_2} \gamma_{i_1 i_2 i_3} y_{i_2} \right| \neq 0 \qquad (i_1, i_3 = 1, \cdots, n).$$

implies that

$C_r$) not every $Y$ is a right hand divisor of zero.

From $C_a$, $C_l$, $C_r$ it follows that

$C_m$) the hypercomplex number system contains a modulus — that is to say, a number $\epsilon = \sum_i \epsilon_i e_i$ exists such that for every $x$

$$\epsilon x = x\epsilon = x;$$

and conversely, from $C_a$, $C_m$ follow $C_l$ and $C_r$.

*We propose to investigate the system of seven conditions:*

$$C_a,\ C_l,\ C_r,\ C_1,\ C_2,\ A,\ B,$$

*and to prove that in this system the conditions $C_1$, $C_2$ are redundant and the remaining five conditions are mutually independent.*

## 2. *Redundancy of $C_1$ and $C_2$.*

I. We regard as given the hypercomplex number system $E \equiv e_1 \cdots e_n$. The units are parted into two sets, $E_j$ and $E_k$,

$$E_j \equiv e_1 \cdots e_m; \qquad E_k = e_{m+1} \cdots e_n$$

and the general number of $E$, namely, $X = x_1 e_1 + \cdots + x_n e_n$, may be considered as the sum of two parts or components

$$X = J + K \qquad \left(J = \sum_j x_j e_j,\ K = \sum_k x_k e_k\right).$$

In this notation the condition $C_1$ states that $J_1 J_2 = J_3$,* viz., the product of any two $J$ numbers is a $J$ number. We proceed to prove that $C_1$ is a consequence of $C_a$, $A$, $B$, $C_l$.†

Setting

$$J_1 J_2 = J_3 + K_3 \tag{1}$$

we prove that

$$XK_3 = 0 \tag{2}$$

for every $X$, and hence, as desired, that $K_3 = 0$, since by $C_l$ not every $X$ is a left hand divisor of 0. We set

$$X = J + K. \tag{3}$$

Then (2) follows at once from the two relations

$$JK_3 = 0, \qquad KK_3 = 0, \tag{4}$$

of which the former holds by $A$, and the latter is evident from (1) by left hand multiplication by $K$ and the use of $C_a$ and $B$.

Hence $C_1$ is a consequence of $C_a$, $C_l$, $A$, $B$.

Similarly, $C_1$ is a consequence of $C_a$, $C_r$, $A$, $B$. For from $C_a$, $A$, $B$ it follows that $K_3 X = 0$ for every $X$.

By parity of reasoning, $C_2$ is a consequence of $C_a$, $A$, $B$ and $C_l$ or $C_r$.

* $J_1 = \sum_j x_{1j} e_j$, $J_2 = \sum_j x_{2j} e_j$, etc.

† By working explicitly with the units the theorem first demonstrated was that $C_1$ is a consequence of $C_a$, $A$, $B$, $C_2$, $C_l$. The suggestion of Professor E. H. MOORE to work immediately with the numbers and their components led me to the discovery that $C_1$ *can be demonstrated without assuming* $C_2$. The method employed for proving the independence of $C_l$ and $C_r$ (§ 5) is also due to Professor MOORE.

Hence, $C_1$ *and* $C_2$ *are consequences of* $C_a$, $A$, $B$, $C_l$ *or* $C_r$, *and a fortiori of* $C_a$, $A$, $B$, $C_m$, that is, *in an associative hypercomplex number system containing a modulus,* $C_1$ *and* $C_2$ *are consequences of* $A$ *and* $B$.

We proceed to prove the mutual independence* of $C_a$, $C_l$, $C_r$, $A$, $B$.

### 3. *Independence of A and B.*

Consider the irreducible system† whose multiplication table is

$E$:

| | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|
| $e_1$ | 0 | 0 | $e_1$ |
| $e_2$ | $e_1$ | $e_2$ | 0. |
| $e_3$ | 0 | 0 | $e_3$ |

Here we have $E \equiv E_j E_k \equiv (e_1 e_2)(e_3)$. The condition $C_a$ is fulfilled, and, as $E$ contains a modulus, namely, $\epsilon = e_2 + e_3$, the conditions $C_l$ and $C_r$ are both fulfilled.

Since $e_3 e_1 = e_3 e_2 = 0$ the condition $B$ (viz.: $e_k e_j = 0$) is also fulfilled. But $e_j e_k$ does not vanish for every $j$ and $k$, for example $e_1 e_3 = e_1 (\neq 0)$. Thus $C_a$, $C_l$, $C_r$, $B$ are satisfied and $A$ is contradicted. Hence $A$ is independent of $C_a$, $C_l$, $C_r$. $B$.

By interchanging $j$ and $k$ one sees that $B$ is independent of $C_a$, $C_l$, $C_r$, $A$.

### 4. *Independence of* $C_a$.

Consider the set of units $e_1$, $e_2$ whose multiplication table is

| | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $e_2$ | 0. |
| $e_2$ | 0 | $e_1$ |

The determinants $\Delta_x$ and $\Delta'_y$ are

$$\Delta_x \equiv \begin{vmatrix} 0 & x_2 \\ x_1 & 0 \end{vmatrix}, \qquad \Delta'_y \equiv \begin{vmatrix} 0 & y_2 \\ y_1 & 0 \end{vmatrix}$$

and do not vanish identically; therefore the conditions $C_l$ and $C_r$ are fulfilled.

Since $e_1 e_2 = e_2 e_1 = 0$ the conditions $A$ and $B$ are also fulfilled. But the condition $C_a$ is contradicted, since

* That $C_a$, $C_l$, $C_r$ are independent was proved by Professor L. E. DICKSON in these Transactions, vol. 4 (1903), pp. 21–24.

† SCHEFFERS, l. c., p. 343.

$$(e_1 e_1) e_2 = e_2 \cdot e_2 = e_1 \neq e_1 (e_1 e_2) = e_1 \cdot 0 = 0 .$$

Hence $C_a$ is independent of $C_r$, $C_l$, $A$, $B$.

### 5. *Independence of $C_l$ and $C_r$.*

I. Consider the two systems

$$E'_j: \quad \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & 0 & e_1 \\ e_2 & 0 & e_2 \end{array}, \qquad E'_k: \quad \begin{array}{c|cc} & e_3 & e_4 \\ \hline e_3 & 0 & e_3 \\ e_4 & 0 & e_4 \end{array}.$$

It is easily verified that in each system $C_a$ is fulfilled. Therefore the system $E' = E'_j E'_k$:

$$E': \quad \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline e_1 & 0 & e_1 & 0 & 0 \\ e_2 & 0 & e_2 & 0 & 0 \\ e_3 & 0 & 0 & 0 & e_3 \\ e_4 & 0 & 0 & 0 & e_4 \end{array},$$

also satisfies the condition $C_a$. The conditions $A$ and $B$ are satisfied in $E'$.

For this system

$$\Delta_x \equiv \begin{vmatrix} 0 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & x_4 \end{vmatrix} \equiv 0, \qquad \Delta'_y \equiv \begin{vmatrix} y_2 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_4 & 0 \\ 0 & 0 & 0 & y_4 \end{vmatrix} \not\equiv 0;$$

thus the condition $C_r$ is fulfilled and $C_l$ is not fulfilled.

Therefore $C_l$ is independent of $C_a$, $C_r$, $A$, $B$.

II. By interchanging *right* and *left* we see at once that $C_r$ is independent of $C_a$, $C_l$, $A$, $B$.

THE UNIVERSITY OF CHICAGO,
*November*, 1903.

# A SIMPLE PROOF OF A THEOREM IN THE CALCULUS OF VARIATIONS

## (EXTRACT FROM A LETTER TO MR. W. F. OSGOOD)*

BY

E. GOURSAT

*Introductory note.*—The proof of the fundamental theorem of my paper entitled: *On a fundamental property of a minimum in the calculus of variations and the proof of a theorem of Weierstrass's*, Transactions, vol. 2 (1901), p. 273, was simplified by BOLZA, ibid., p. 422. In a letter bearing the date June 21, 1903, Professor GOURSAT has communicated to me a new proof of this theorem, which for simplicity and directness leaves nothing to be desired. With his permission an extract from his letter, containing this proof, is here given.

W. F. OSGOOD.

En lisant votre mémoire sur le calcul des variations, dans les Transactions of the American Mathematical Society (vol. 2, 1901), il m'a semblé que la démonstration du théorème fondamental de la page 286 pouvait être simplifiée comme il suit.

La fonction $f(x)$ étant continue dans l'intervalle $(a, b)$, où $a < b$, et admettant une dérivée $f'(x)$ continue dans cet intervalle, il s'agit de trouver une limite inférieure de l'intégrale definie

$$\int_a^l f'(x)^2 dx, \quad a < l \leqq b.$$

Je m'appuie pour cela sur la remarque élémentaire suivante. Soient

$$\alpha_1, \alpha_2, \cdots, \alpha_n,$$

$$h_1, h_2, \cdots, h_n,$$

deux suites de $n$ nombres, les $h_i$ étant tous positifs. On a toujours

$$\frac{\alpha_1^2}{h_1} + \frac{\alpha_2^2}{h_2} + \cdots + \frac{\alpha_n^2}{h_n} \geqq \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)^2}{h_1 + h_2 + \cdots + h_n}.$$

* Presented to the Society October 31, 1903. Received for publication September 23, 1903.

On le vérifie immédiatement si $n=2$, et on l'étend ensuite de proche en proche. Cela posé, decomposons l'intervalle $(a, l)$ en intervalles partiels

$$a, x_1, x_2, \cdots, x_{n-1}, l$$

et soit

$$h_1 = x_1 - a, \qquad h_2 = x_2 - x_1, \cdots, h_n = l - x_{n-1}.$$

On a respectivement

$$\begin{aligned} f(x_1) - f(a) &= h_1 f'(\xi_1), & a < \xi_1 < x_1, \\ f(x_2) - f(x_1) &= h_2 f'(\xi_2), & x_1 < \xi_2 < x_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f(l) - f(x_{n-1}) &= h_n f'(\xi_n), & x_{n-1} < \xi_n < l. \end{aligned}$$

On en tire

$$[f'(\xi_1)]^2 h_1 + [f'(\xi_2)]^2 h_2 + \cdots + [f'(\xi_n)]^2 h_n$$
$$= \frac{[f(x_1) - f(a)]^2}{h_1} + \frac{[f(x_2) - f(x_1)]^2}{h_2} + \cdots + \frac{[f(l) - f(x_{n-1})]^2}{h_n},$$

et le second membre est d'aprés la remarque faite plus haut,

$$\geqq \frac{[f(l) - f(a)]^2}{l - a},$$

ou

$$\geqq \frac{L^2}{l - a},$$

L désignant, d'aprés votre notation, $|f(l) - f(a)|$. En faisant croître $n$ indéfiniment, on a donc à la limite

$$\int_a^l [f'(x)]^2 dx \geqq \frac{L^2}{l - a}. \tag{1}$$

L'inégalité

$$\int_a^l [f'(x)]^2 dx \geqq \frac{L^3}{54(b-a)^2} \tag{2}$$

sera assurée si l'on a

$$\frac{L^3}{54(b-a)^2} < \frac{L^2}{l - a},$$

ou

$$L < \frac{54(b-a)^2}{l - a}.$$

Il en sera ainsi toutes les fois que $L$ est $< 3\sqrt{2}(b-a)$, puisque l'on a

$$3\sqrt{2}(b-a) < \frac{54(b-a)^2}{l-a},$$

lorsque

$$a < l \leqq b.$$

Il serait facile de généraliser le théorème, si cela pouvait être de quelque utilité.